A SUPER VERSION OF ZHU'S THEOREM

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2008

University of Oregon Graduate School

Confirmation of Approval and Acceptance of Dissertation prepared by:

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Title:

"A Super Version of Zhu's Theorem"

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June 14, 2008

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An Abstract of the Dissertation of

Alex Jordan for the degree of Doctor of Philosophy

in the Department of Mathematics to be taken June 2008

Title: A SUPER VERSION OF ZHU'S THEOREM

Approved: ______________________________

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We generalize a theorem of Zhu relating the trace of certain vertex algebra representations and modular invariants to the arena of vertex super algebras. The theorem explains why the space of simple characters for the Neveu-Schwarz minimal models $\mathrm{NS}(p, q)$ is modular invariant. It also expresses negative products in terms of positive products, which are easier to compute. As a consequence of the main theorem, the subleading coefficient of the singular vectors of $\mathrm{NS}(p, q)$ is determined for p and q odd. An interesting family of q-series identities is established. These consequences established here generalize results of Milas in this field.

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ACKNOWLEDGMENTS

I thank my family for enabling me to accomplish so much, my advisor for introducing me to this problem, the U of O math department and the GTFF for giving me wonderful teaching experience, and all my friends in Eugene who helped to make these past six years so memorable.

To my grandpa Irvine

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CHAPTER I

PRELIMINARIES

In this chapter we recall some preliminary information about vertex algebras and their representations, including Zhu's Theorem. The main result of this paper is an augmentation to Zhu's Theorem that makes it applicable to super algebras.

For ease of reference the basics of vertex algebra theory are laid out here, however it is assumed that the reader is familiar with this information, which is the topic of introductory texts [6] and [5].

I.1 Vertex Algebras and the Virasoro Algebra

The definition of a vertex algebra is complicated, being motivated by theoretical small-scale physics. For the lay-mathematician, there are some notable features separating vertex algebras from other algebras. In place of a single bilinear multiplication, there are infinitely many products for any two vectors a and b. These products are indexed by $\mathbb{Z}$ and are written $a_n b$. Rather than studying the nth multiplication by a individually, we study all products with a at once in the form of the generating series $\sum a_n z^{-n-1}$. In physics, this series is thought of as a meromorphic expansion of a quantum field in End$V$ around a point on a Riemann surface with local coordinate z ([12]).

In [2], some four or five equivalent definitions for vertex algebras are proposed. We present here the defintion of [6].

Definition I.1. A *vertex algebra* consists of the following data:

- a graded vector space V
- a distinguished vector $\mathbf{1} \in V_0$
- the *state-field correspondence* $\mathrm{Y} : V \to \mathrm{End}V[[z, z^{-1}]]$, with conventional notation $\mathrm{Y}(a, z) = \sum_{\mathbb{Z}} a_n z^{-n-1}$. (So a_n is in $\mathrm{End}V$.) Even more restrictive, $Y(a)$ must be a *field*. That is, for any $b \in V$ there is an integer N with $a_n b = 0$ for $n > N$. Finally, Y should respect degree; a_n must be a degree $\deg a - n - 1$ endomorphism.

subject to the following axioms:

- For all $a \in V$, $\mathrm{Y}(a, z)\mathbf{1} \in az^0 + zV[[z]]$ and $\mathrm{Y}(\mathbf{1}, z) = \mathrm{Id}\, z^0$. (*vacuum* or *identity* axiom)
- Define a degree 1 homogeneous operator T by $Ta = a_{-2}\mathbf{1}$. For all $a \in V$, $[T, Y(a, z)] = \partial_z \mathrm{Y}(a, z)$. (*translation covariance*)
- For all a, $b \in V$, $(z - w)^N [\mathrm{Y}(a, z), \mathrm{Y}(b, w)] = 0$ for large enough N. (*locality*)

The vacuum axiom provides for a vacuous state whose field interacts trivially with other fields. The translation operator and the translation axiom provide a structured way to boost states one level higher. Locality is a reasonable generalization for the commutativity of two fields.

Two consequences of the axioms will be used in the proof of the main theorem, so we present them here. These are proved in [5].

Proposition I.1. *Associativity.*

For any a, b in a vertex algebra V,

$$\begin{aligned}&\mathrm{Res}_{w-z}(w-z)^i Y(Y(b,w-z)a,z)\\ &= \mathrm{Res}_w\, \iota_{w,z}(w-z)^i Y(b,w)Y(a,z) - \mathrm{Res}_w\, \iota_{z,w}(w-z)^i Y(a,z)Y(b,w).\end{aligned}$$

Here, $\iota_{w,z}$ is the expansion of a rational series in w and z in the domain with $|w| > |z|$.

Proposition I.2. *Borcherd's Identity.*

For any a, b in a vertex algebra V,

$$[b_l, a_k] = \sum_{i=0}^{\infty} \binom{l}{i} (b_i a)_{l+k-i}.$$

Consider the ring of complex-valued C^∞ functions on the circle $\mathcal{S}^1$. All such functions have Fourier expansions $\sum_{\mathbb{Z}} c_n z^n$ in the complex unit circle coordinate z. Some such expressions in $\mathbb{C}[[z, z^{-1}]]$ converge, and some do not. We consider finite sums, where convergence is not an issue. So our attention is restricted to $\mathbb{C}[z, z^{-1}]$ and its Lie algebra of derivations $\mathcal{D} = \mathrm{Der}\,(\mathbb{C}[z, z^{-1}])$. The Lie algebra $\mathcal{D}$ has a basis consisting of the Virasoro elements $L_n = -z^{n+1}\partial_z$. In $\mathcal{D}$, the Virasoro elements satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{I.1}$$

The Virasoro Lie algebra Vir is the unique nontrivial central extension of $\mathcal{D}$ by the one-dimensional space $\mathbb{C}C$, with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C. \tag{I.2}$$

(The cocycle $\frac{m^3-m}{12}$ is a traditional choice - all other choices give isomorphic or trivial extensions.) Vir is of essential importance in defining a conformal vertex algebra.

Definition I.2. A *conformal vertex algebra* is a vertex algebra V containing a distinguished vector $\omega \in V_2$ (called a *conformal vector*) such that

(a) Under the identification $\omega_n = L_{n-1}$, the ω_n satisfy the relations I.2, with C acting as multiplication by a complex number c.

(b) ω_1 (or L_0) is the grading operator for V.

(c) ω_0 (or L_{-1}) is the translation operator for V.

The number c is called the *central charge* of V.

The first example of a conformal vertex algebra comes directly from Vir itself. Vir has the $\mathbb{Z}$-grading with $\deg(L_m) = -m$ and $\deg(C) = 0$. We allow $\mathrm{Vir}_{\leq 0}$ to act on $\mathbb{C}$ in an almost trivial manner, and then induce the action to Vir. Choose two numbers c and h, and define an action of $\mathrm{Vir}_{\leq 0}$ on $\mathbb{C}\mathbf{1}$ by

$$\begin{aligned} C\mathbf{1} &= c\mathbf{1} \\ L_0\mathbf{1} &= h\mathbf{1} \end{aligned}$$

and $L_m\mathbf{1} = 0$ for $m > 0$. Define V_c^h to be the minimal quotient of the Vir module $\mathrm{Vir} \otimes_{\mathrm{Vir}_{\leq 0}} \mathbb{C}\mathbf{1}$. For certain choices of c, $V_c = V_c^0$ has the structure of a *rational* conformal vertex algebra, that is, one with finitely many simple representations. The finite list of simple representations can be given formulaically as a list of V_c^h

for certain h. All of this is established in [4] and [11]. Specifically, this happens when

$$\begin{aligned} c &= 1 - \frac{6(p-q)}{pq} \\ h &= \frac{(pr-qs)^2 - (p-q)^2}{4pq} \end{aligned}$$

where p and q are integers greater than 1, and r (resp. s) ranges between 1 and $q-1$ (resp. 1 and $p-1$).

I.2 A Change of Coordinates for Vertex Algebras

Zhu recognized in [12] that there is a change of coordinates we may apply to the fields in the state-field correspondence of a vertex algebra V. If the given state field correspondence is denoted by $Y(a,z) = \sum a_n z^{-n-1}$, let $Y[a,z]$ be defined by

$$\begin{aligned} Y[a,z] &= e^{d_a z} Y(a, e^z - 1) && \text{(I.3)} \\ &= \sum a[n] z^{n-1} && \text{(I.4)} \end{aligned}$$

Geometrically, $Y(\ ,z)$ uses z along the complex unit circle to coordinatize a string. $Y[\ ,z]$ uses z along a much less standard cirlce - one that passes through $\ln(2)$, $\ln(1+i)$, $-\infty$, and $\ln(1-i)$.

It is known (see [12]) that V with the state-field correspondence $Y(\ ,z)$ is isomorphic as a vertex algebra to V with the state-filed correspondence $Y[\ ,z]$. The isomorphism does not change the vacuum vector $\mathbf{1}$. However, the two vertex algebras have different conformal vectors. If ω is the conformal vector for V with $Y(\ ,z)$, then $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$ is the conformal vector for V with $Y[\ ,z]$.

Define $c(d,i,m)$ by $\binom{d-1+x}{i} = \sum_{m=0}^{i} c(d,i,m)x^m$. Then the $a[m]$ may be expressed in terms of the a_i:

$$a[m] = m! \sum_{i=s}^{\infty} c(d,i,m)a_n \tag{I.5}$$

This formula will be used in the proof of the main theorem, and is proved in [12].

I.3 Modularity

A basic familiarity with the theory of the modular group $SL_2(\mathbb{Z})$ and its action on $\mathbb{C}$ is useful to understand the importance of Zhu's theorem and the main theorem. The texts [8] and [3] are good introductions to the subject. The action of $\mathrm{SL}_2(\mathbb{Z})$ on the complex plane may be shortly described as follows. Identify $\mathbb{C} \cup \{\infty\}$ with the projective space $\mathbb{CP}^1$ according to

$$\begin{aligned} \tau &\leftrightarrow \binom{\tau}{1} \\ \infty &\leftrightarrow \binom{1}{0}. \end{aligned}$$

Under this identification, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C} \cup \{\infty\}$ via matrix multiplication. This action clearly preserves $\mathbb{R} \cup \{\infty\}$, and the positive determinant of any matrix γ ensures that the upper half plane is mapped to itself as well.

A meromorphic function on $\mathbb{C}$ is called a *modular function of weight* k if $f(\gamma\tau) = \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \gamma \binom{\tau}{1}\right)^k f(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$. For subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$, we can have *modular functions of weight k for* Γ defined in the exact same way. Some important modular functions are given by the (normalized) Eisenstein series:

$$\tilde{G}_{2k}(q) = \frac{-B_{2k}}{(2k)!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}$$

for $k \geq 1$where $q = e^{2\pi i\tau}$ and the B_k are the Bernoulli numbers. The weight of $\tilde{G}_{2k}$ is $2k$. These series appear in Zhu's theorem below, which demonstrates the modularity of conformal vertex algebra traces.

The following series appear in the main theorem:

$$Q^+_{2k+1}(q) = \frac{B_{2k+1,d}}{(2k+1)!} - \frac{2}{(2k+1)!}\sum_{n=0}^{\infty}\frac{(n+1/2)^{2k+1}q^{n+1/2}}{1+q^{n+1/2}}$$

for $k \geq 0$ and where $B_{m,d}$ are defined for $m \geq 1$ and $d \in \frac{1}{2} + \mathbb{Z}$ by the conditionally convergent double series

$$B_{m,d} = \sum_{i=d-1/2}^{\infty}\sum_{n=0}^{i-d+1/2}(-1)^{i-d-n+1/2}c_{d,i}\binom{i}{d+n-1/2}(n+1/2)^m.$$

The $c_{d,i}$ in the above series are defined by $(1+z)^{d-1}\ln(1+z)^{-1} = \sum c_{d,i}z^i$. For convenience, we set $Q^+_{2k}(q) = \frac{B_{2k,d}}{(2k)!}$. (Although we hypothesize that these numbers are 0.) The series Q^+_k appear in the super version of Zhu's theorem that is our main result.

There are two more series with modular properties important to us:

$$\eta(q) = \prod_{n=1}^{\infty}(1-q^n)$$

$$\eta_0(q) = \prod_{n=1}^{\infty}\frac{1+q^{n-1/2}}{1-q^n}$$

The series $q\,\eta(q)^{24}$ is famously a modular form of wieght 12, also known as the discriminant modular form $\Delta(q)$.

I.4 Trace, Zhu's Theorem, and the work of Milas

Starting with any space M and a diagonalizable operator L whose eigenspaces are finite dimensional, then given any operator B we can define its q-trace. The idea is to collect the trace data for the restriction-projection of B to each of L's eigenspaces. If the eigenvalues of L run through $\mathbf{Z}$ for example, then B has restriction-projections B_n for $n \in \mathbf{Z}$, and we write:

$$\operatorname{tr}|_M Bq^L = \sum_{n\in\mathbf{Z}} \operatorname{tr} B_n q^n \tag{I.6}$$

The first example of a q-trace would be the case when B is the identity map. Then $\operatorname{tr} B_n$ is just the dimension of the n-eigenspace, and the q-trace of B is just the character of M. Standard trace has the commutativity property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, and q-trace has an analogous commutativity property. If A is of degree k and B of degree $-k$, then $\operatorname{tr}|_M ABq^L = q^k \operatorname{tr}|_M BAq^L$.

In a vertex algebra V, consider a homogeneous vector v. Then only one of the v_n has degree 0 as an endomorphism of V, and therefore this is the only v_n that could have non-zero trace. In light of this Zhu introduced the notation $\mathrm{o}(v)$ to mean $v_{\deg(v)-1}$, which is the one v_n that might have interesting trace. We emphasize that $\mathrm{o}(v) = v_{\deg(v)-1}$, and not $v[\deg(v)-1]$. Zhu proved the following theorem in [12].

Theorem I.1. *Zhu's Theorem*

Let a and b be homogeneous vectors in a conformal vertex algebra V with representation M. Then

$$\begin{aligned} \operatorname{tr}|_M \operatorname{o}(b[0]a)q^{L_0} &= 0 \\ \operatorname{tr}|_M \operatorname{o}(b[-1]a)q^{L_0} &= \operatorname{tr}|_M \operatorname{o}(b)\operatorname{o}(a)q^{L_0} + \sum_{k\geq 1} \tilde{G}_{2k}(q)\operatorname{tr}|_M \operatorname{o}(b[2k-1]a)q^{L_0}. \end{aligned}$$

This theorem is the basis for proving the modular invariance of the span of V-module characters.

In [11] the characters of the rational vertex algebra V_c^0 and of its simple representations V_c^h were determined. We do not present them here, since we later present these characters for the corresponding super algebras which are of more importance to this text. In [?, ?, Milas] Milas discovered that he could apply Zhu's Theorem to the singular vector of V_c^0, with M ranging through the modules V_c^h. Given the knowledge of characters provided by [11] and some basic differential equation theory, he proved anew a family of Ramanjan-style q-series identities and proved modular invariance of the product of V_c^h-characters.

Noting that [11] had also established characters for some vertex super algebras, we set out to extend Milas's work. The first obstacle was that Zhu's theorem was inapplicable when odd vectors were in play. That led to the main result of this paper, an augmentation that deals with odd vectors. Once this was established, we could continue in the style of Milas searching for Ramanujan-style identities. We also learned information about the singular vector in Neveu-Schwarz algebras.

CHAPTER II

VERTEX SUPER ALGEBRAS

In this chapter we discuss the super analogs of the fundamentals discussed in the introduction. Although I make an attempt to motivate the topics, a full explanation of what follows would obscure the main result.

II.1 The Neveu-Schwarz Algebra

The Neveu-Shwarz Lie algebra NS is of fundamental importance to us. It is the super version of Vir. The idea is to adjoin a square root of each L_m to Vir, i.e. we would like to have $G_{\frac{m}{2}}$ in the algebra with $G_{\frac{m}{2}}G_{\frac{m}{2}} = L_m$. But in a Lie algebra these kinds of products make no sense. If we allow these $G_{\frac{m}{2}}$ to be odd, then two problems are solved at once. First, the relation we are looking for becomes $\left[G_{\frac{m}{2}}, G_{\frac{m}{2}}\right] = 2L_m$ in the universal enveloping algebra. Second, this suggests a geometric motivation as to from where such an algebra might naturally arise.

The super circle $S^{1|1}$ is an object of study in theoretical physics - see [7]. By definition, it is the topological space S^1 together with a super algebra denoted $C^\infty(S^{1|1})$. This super algebra is an extension of the familiar $C^\infty(S^1)$, acheived by tensoring with the exterior algebra on one odd variable ζ. That is $C^\infty(S^{1|1}) = C^\infty(S^1) \otimes \Lambda(\zeta)$. This defines a ring of super functions on $S^{1|1}$.

Consider derivations on the super circle, $\mathrm{Der}(C^\infty(S^{1|1}))$. The set of such things is topologically spanned by derivations of the form $z^n\partial_z$, $z^n\partial_\zeta$, $z^n\zeta\partial_z$, and $z^n\zeta\partial_\zeta$. As before, we first restrict our attention to finite sums from the canonical topological basis. As a $\mathbb{C}[z, z^{-1}]$-module, this object is 4-dimensional. There is a 2-dimensional subalgebra we are interested in. For $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, we define:

$$L_n = -z^{n+1}\partial_z - \frac{n+1}{2}z^n\zeta\partial_\zeta \tag{II.1}$$

$$G_r = iz^{r+\frac{1}{2}}(\zeta\partial_z + \partial_\zeta) \tag{II.2}$$

(Note we will only have square roots G_r for L_n with n odd. The algebra with square roots of the other L_n is called the Ramond algebra.) Any $\mathcal{D}$-module is also a representation of $\mathrm{Der}(C^\infty(S^{1|1}))$, whereby ζ and ∂_ζ act as 0. As such, the L_n we've just defined project to the L_n in $\mathcal{D}$.

Let's denote the Lie algebra spanned by the L_n and the G_r by $\overline{\mathcal{D}}$. In $\overline{\mathcal{D}}$ we have the commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} \tag{II.3}$$

$$[G_r, L_m] = (r - \frac{m}{2})G_{r+m} \tag{II.4}$$

$$[G_r, G_s] = 2L_{r+s} \tag{II.5}$$

We define the Neveu-Scwarz algebra NS to be the one-dimensional central extension of $\overline{\mathcal{D}}$ with relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C \tag{II.6}$$

$$[G_r, L_m] = \left(r - \frac{m}{2}\right) G_{r+m} \tag{II.7}$$

$$[G_r, G_s] = 2L_{r+s} + \delta_{r+s,0}\frac{4r^2-1}{12}C. \tag{II.8}$$

II.2 Super Conformal Vertex Algebras and Representations of NS

Just as the definition of a conformal vertex algebra centered around the algebra Vir, the definition of a conformal vertex super algebra centers around NS.

Definition II.1. A *conformal vertex super algebra* is a graded vertex super algebra V containing a distinguished vector $\tau \in V_{3/2}$ (called a *superconformal vector*) and a conformal vector ω equal to $\tau_{(0)}\tau$ such that

(a) Under the identifications $\tau_n = G_{n-1/2}$ and $\omega_n = L_{n-1}$, the τ_n and ω_n satisfy the relations II.6 - II.8 with C acting as multiplication by a complex number c.

(b) ω_1 (or L_0) is the grading operator for V.

(c) ω_{-1} (or L_{-2}) is the translation operator for V.

The number c is called the *central charge* of V.

Most of what follows in this section is established in [4] and [11]. NS has the obvious $\frac{1}{2}\mathbb{Z}$-grading with $\deg(L_m) = -m$, $\deg(G_r) = -r$, and $\deg(C) = 0$. If we allow $\mathrm{NS}_{\leq 0}$ to act in a simple way on $\mathbb{C}$ and induce to all of NS, we get an important NS representation. Specifically, choose a $c \in \mathbb{C}$. Let $\mathbb{C}\mathbf{1}$ be a $\mathrm{NS}_{\leq 0}$-module where L_m and G_r act trivially for $m, r \geq 0$ and $C \cdot \mathbf{1} = c\mathbf{1}$. Then define the Verma module:

$$V_c = \mathrm{NS} \otimes_{\mathrm{NS}_{\leq 0}} \mathbb{C}\mathbf{1} \tag{II.9}$$

This NS-module has a unique minimal quotient which we call NS(c). It will be important to us that this module NS(c) has the structure of a vertex super algebra (see [11]). The superconformal vector τ is taken to be $G_{-3/2}\mathbf{1}$. From now on when referring to NS(c), we will be referring to it as a vertex super algebra, not just an NS-module.

For most values of c, NS(c) has infinitely many irreducible representations. However, there is a two parameter family of c's for which NS(c) has finitely many irreducible representations, all of which can be described. Let

$$c_{p,q} = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right) \tag{II.10}$$

where $q - p \in 2\mathbb{Z}$ and $\gcd\left(\frac{q-p}{2}, p\right) = 1$. These are precisely the c's for which NS(c) has finitely many irreducible representations. From now on, we write
NS(p, q) = NS($c_{p,q}$).

Once p and q are fixed, we define the numbers

$$h^{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{8pq} \tag{II.11}$$

where $1 \leq r < q$, $1 \leq s < p$, and r and s have matching parity. These numbers are called the energies of the irreducible modules of $NS(p,q)$, and they determine its irreducible modules. Specifically, we once again take a representation of $NS_{\leq 0}$ on $\mathbb{C}\mathbf{1}$. This time, $C\mathbf{1} = c_{p,q}\mathbf{1}$, L_m and G_r act trivially for $m, r > 0$, but $L_0\mathbf{1} = h^{r,s}\mathbf{1}$. The Verma module obtained this way,

$$V_c^{h^{r,s}} = NS \otimes_{NS_{\geq 0}} \mathbb{C}\mathbf{1}$$

has a unique minimal quotient as an NS-module, which we call $NS(p,q)^{h^{r,s}}$. These $NS(p,q)^{h^{r,s}}$ are the irreducible representations of $NS(p,q)$.

CHAPTER III

A SUPER VERSION OF ZHU'S THEOREM

In this chapter we follow the consequences of commuting two odd vectors in a vertex superalgebra, and obtain a super version of Zhu's trace identity.

III.1 A Lemma

We need the following lemma.

Lemma III.1. *Let a and b be two odd vectors in the vertex super algebra V, and let M be any V-module. Then*

$$(a)\quad w^{d_b}\,\mathrm{tr}\,|_M Y(a,z)Y(b,w)q^{L_0} = \sum_{m=0}^{\infty} z^{d_b-m-1} Q_m(\frac{z}{w},q)\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0} \tag{III.1}$$

$$(b)\quad w^{d_b}\,\mathrm{tr}\,|_M Y(b,w)Y(a,z)q^{L_0} = \sum_{m=0}^{\infty}(-1)^m z^{d_b-m-1} Q_m(\frac{w}{z},q)\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0} \tag{III.2}$$

where

$$Q_m(x,q) = \frac{1}{m!}\sum_{r\in Z+\frac{1}{2}} x^r \frac{q^r}{1+q^r}(r)^m$$

Proof. For the first part of the lemma, we begin by expanding the state-field correspondence and observing that most terms cannot contribute to the trace.

$$w^{d_b} \operatorname{tr}|_M Y(a,z)Y(b,w)q^{L_0} = w^{d_b} \operatorname{tr}|_M \sum_{k,l} a(k)b(l)z^{-k-1}w^{-l-1}q^{L_0} \quad \text{(III.3)}$$

$$= w^{-d_a+1}z^{-1}\sum_k z^{-k}w^k \operatorname{tr}|_M a(k)b(l)q^{L_0} \quad \text{(III.4)}$$

where $l = d_a + d_b - k - 2$ in the last term (so that the degree of $a(k)b(l)$ is 0.) Keeping in mind that a and b are odd, and using the commutativity of tr,

$$\operatorname{tr}|_M a(k)b(l)q^{L_0} = q^{d_a-k-1} \operatorname{tr}|_M b(l)a(k)q^{L_0} \quad \text{(III.5)}$$

$$= q^{d_a-k-1} \operatorname{tr}|_M \left([b(l),a(k)] - a(k)b(l)\right) q^{L_0} \quad \text{(III.6)}$$

So we may solve for $\operatorname{tr}|_M a(k)b(l)q^{L_0}$ and then apply the Borcherds identity (proposition I.2):

$$\operatorname{tr}|_M a(k)b(l)q^{L_0} = \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} \operatorname{tr}|_M [b(l),a(k)]q^{L_0} \quad \text{(III.7)}$$

$$= \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} \operatorname{tr}|_M \sum_{i=0}^{\infty} \binom{l}{i} \left(b(i)a\right)_{l+k-i} q^{L_0} \quad \text{(III.8)}$$

Combining Eq. (III.4) with Eq. (III.8), we have:

$$w^{d_b} \operatorname{tr}|_M Y(a,z)Y(b,w)q^{L_0}$$

$$= \left(\frac{w}{z}\right)^{1-d_a} z^{-d_a} \sum_k \left(\frac{w}{z}\right)^k \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} \operatorname{tr}|_M \sum_{i=0}^{\infty} \binom{l}{i} \left(b(i)a\right)_{l+k-i} q^{L_0} \quad \text{(III.9)}$$

$$= \operatorname{tr}|_M \left(\left(\frac{w}{z}\right)^{1-d_a} z^{-d_a} \sum_k \sum_{i=0}^{\infty} \left(\frac{w}{z}\right)^k \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} \binom{l}{i} \left(b(i)a\right)_{l+k-i} \right) q^{L_0} \quad \text{(III.10)}$$

$$= \operatorname{tr}|_M \left(\left(\frac{w}{z}\right)^{1-d_a} z^{-d_a} \sum_k \sum_{i=0}^{\infty} \left(\frac{w}{z}\right)^k \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} \sum_{m=0}^{i} c(d_b,i,m)(d_a-k-1)^m \left(b(i)a\right)_{l+k-i} \right) q^{L_0}$$
$$= \operatorname{tr}|_M \left(\left(\frac{w}{z}\right)^{1-d_a} z^{-d_a} \sum_k \sum_{m=0}^{\infty} \left(\frac{w}{z}\right)^k \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} (d_a-k-1)^m \sum_{i=m}^{\infty} c(d_b,i,m) \left(b(i)a\right)_{l+k-i} \right) q^{L_0}$$
$$= \operatorname{tr}|_M \left(\left(\frac{w}{z}\right)^{1-d_a} z^{-d_a} \sum_k \sum_{m=0}^{\infty} \left(\frac{w}{z}\right)^k \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} (d_a-k-1)^m \sum_{i=m}^{\infty} c(d_b,i,m) \circ \left(b(i)a\right) \right) q^{L_0}$$

Recalling Eq. (I.5), we have:

$$\begin{aligned}
& w^{d_b} \operatorname{tr}|_M Y(a,z)Y(b,w)q^{L_0} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_k \left(\frac{z}{w}\right)^{d_a-k-1} \frac{q^{d_a-k-1}}{1+q^{d_a-k-1}} (d_a-k-1)^m \right) z^{-d_a} \operatorname{tr}|_M \circ (b[m]a)\, q^{L_0} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{r \in Z+\frac{1}{2}} \left(\frac{z}{w}\right)^r \frac{q^r}{1+q^r} (r)^m \right) z^{d_b-m-1} \operatorname{tr}|_M Y\left(b[m]a, z\right)\right) q^{L_0} \\
&= \sum_{m=0}^{\infty} Q_m\left(\frac{z}{w}, q\right) z^{d_b-m-1} \operatorname{tr}|_M Y\left(b[m]a, z\right)\right) q^{L_0}
\end{aligned}$$

This proves part (a) of the lemma. For the second part, we swap the roles of (a, z) and k with (b, w) and l. In Eq. (III.7), since a and b are odd, $[a(k), b(l)] = [b(l), a(k)]$. Many of the terms in Eq. (III.9) remain unchanged, despite the swapping of roles. We have:

$$\begin{aligned}
& w^{d_b} \operatorname{tr}|_M Y(b,w)Y(a,z)q^{L_0} \\
&= \left(\frac{z}{w}\right)^{1-d_b} z^{-d_a} \sum_l \left(\frac{z}{w}\right)^l \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} \operatorname{tr}|_M \sum_{i=0}^{\infty} \binom{l}{i} \left(b(i)a\right)_{l+k-i} q^{L_0} \\
&= \operatorname{tr}|_M \left(\left(\frac{z}{w}\right)^{1-d_b} z^{-d_a} \sum_l \sum_{i=0}^{\infty} \left(\frac{z}{w}\right)^l \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} \binom{l}{i} \left(b(i)a\right)_{l+k-i} \right) q^{L_0}
\end{aligned}$$

$$= \mathrm{tr}|_M \left(\left(\frac{z}{w}\right)^{1-d_b} z^{-d_a} \sum_l \sum_{i=0}^{\infty} \left(\frac{z}{w}\right)^l \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} \sum_{m=0}^{i} c(d_b,i,m)(d_a-k-1)^m (b(i)a)_{l+k-i} \right) q^{L_0}$$

$$= \mathrm{tr}|_M \left(\left(\frac{z}{w}\right)^{1-d_b} z^{-d_a} \sum_l \sum_{m=0}^{\infty} \left(\frac{z}{w}\right)^l \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} (d_a-k-1)^m \sum_{i=m}^{\infty} c(d_b,i,m)(b(i)a)_{l+k-i} \right) q^{L_0}$$

$$= \mathrm{tr}|_M \left(\left(\frac{z}{w}\right)^{1-d_b} z^{-d_a} \sum_l \sum_{m=0}^{\infty} \left(\frac{z}{w}\right)^l \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} (-(d_b-l-1))^m \sum_{i=m}^{\infty} c(d_b,i,m) \circ (b(i)a) \right) q^{L_0}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_l \left(\frac{w}{z}\right)^{d_b-l-1} \frac{q^{d_b-l-1}}{1+q^{d_b-l-1}} (d_b-l-1)^m (-1)^m \right) z^{-d_a}\, \mathrm{tr}|_M \circ (b[m]a)\, q^{L_0}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \left(\sum_{r\in Z+\frac{1}{2}} \left(\frac{w}{z}\right)^r \frac{q^r}{1+q^r} (r)^m \right) z^{d_b-m-1}\, \mathrm{tr}|_M Y\left(b[m]a, z\right) q^{L_0}$$

$$= \sum_{m=0}^{\infty} (-1)^m Q_m\left(\frac{w}{z}, q\right) z^{d_b-m-1}\, \mathrm{tr}|_M Y\left(b[m]a, z\right) q^{L_0}.$$

□

III.2 The Main Theorem

Now we are ready to prove the main theorem.

Theorem III.1. *Let a and b be two odd vectors in the vertex super algebra V, and let M be any V-module. Then*

$$\mathrm{tr}|_M \circ (b[-1]a)\, q^{L_0} = \sum_{m=1}^{\infty} Q_m^+(q)\, \mathrm{tr}|_M \circ (b[m]a)\, q^{L_0}$$

Proof. Recall the numbers c_i, defined by the equation

$$(1+z)^{d_b-1} \ln(1+z)^{-1} = \sum_{i=-1}^{\infty} c_i z^i. \tag{III.11}$$

Then we may write:

$$
\begin{aligned}
\operatorname{tr}|_M \, o\,(b[-1]a)\, q^{L_0} &= \sum_{i=-1}^{\infty} c_i \operatorname{tr}|_M \, o\,(b(i)a)\, q^{L_0} \qquad \text{(III.12)}\\
&= \sum_{i=-1}^{\infty} c_i \operatorname{tr}|_M z^{d_a+d_b-i-1} Y\,(b(i)a, z)\, q^{L_0}\\
&= \sum_{i=-1}^{\infty} c_i z^{d_a+d_b-i-1} \operatorname{Res}_{w-z}(w-z)^i \operatorname{tr}|_M Y\,(Y\,(b, w-z)\, a, z)\, q^{L_0}
\end{aligned}
$$

According to associativity for vertex super algebras (the super version of proposition I.1), this equals

$$
\begin{aligned}
&\sum_{i=-1}^{\infty} c_i z^{d_a+d_b-i-1}\Big(\operatorname{Res}_w \iota_{w,z}(w-z)^i \operatorname{tr}|_M Y(b,w)Y(a,z)q^{L_0} + \operatorname{Res}_w \iota_{z,w}(w-z)^i \operatorname{tr}|_M Y(a,z)Y(b,w)q^{L_0}\Big)\\
&= \sum_{i=-1}^{\infty} c_i z^{d_a+d_b-i-1}\Big(\operatorname{Res}_w \iota_{w,z}(w-z)^i w^{-d_b}\sum_{m=0}^{\infty}(-1)^m Q_m\left(\frac{w}{z},q\right)\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}\\
&\qquad + \operatorname{Res}_w \iota_{z,w}(w-z)^i w^{-d_b}\sum_{m=0}^{\infty} Q_m\left(\frac{z}{w},q\right)\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}\Big).
\end{aligned}
$$

Here we have used lemma III.1. By $\iota_{z,w}(z-w)^i$ we mean the power series expansion of $(z-w)^i$ in the domain where $|z|>|w|$. For most i In the expression above, $\iota_{z,w}(z-w)^i$ is simply $(z-w)^i$. The exception is when $i=-1$. So we will compute the $i=-1$ term of the sum separately. Note $c_{-1}=1$. We find the $i=-1$ term to be:

$$
\begin{aligned}
&z^{d_a+d_b}\Big(\operatorname{Res}_w \sum_{n=1}^{\infty} w^{-n}z^{n-1}w^{-d_b}\sum_{m=0}^{\infty}\frac{1}{m!}(-1)^m \sum_{r\in\mathbb{Z}+\frac{1}{2}} w^r z^{-r}\frac{r^m q^r}{1+q^r}\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}+\\
&\quad \operatorname{Res}_w \sum_{n=0}^{\infty} -w^n z^{-n-1}w^{-d_b}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r\in\mathbb{Z}+\frac{1}{2}} w^{-r}z^{r}\frac{r^m q^r}{1+q^r}\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}\Big)\\
&= z^{d_a}\Big(-\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{\prime}\frac{r^m q^r}{1+q^r}\operatorname{tr}|_M Y(b[m]a,z)q^{L_0} + \sum_{m=0}^{\infty}(-1)^m\sum_{r=d_b}^{\prime}\frac{r^m q^r}{1+q^r}\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}\Big)\\
&= -z^{d_a}\sum_{m=0}^{\infty}\Big(2\sum_{r=d_b}^{\prime}\frac{r^{2m+1}q^r}{1+q^r}\frac{1}{(2m+1)!}\operatorname{tr}|_M Y(b[2m+1]a,z)q^{L_0} + \sum_{r=-d_b+1}^{d_b-1}\frac{r^m q^r}{1+q^r}\frac{1}{m!}\operatorname{tr}|_M Y(b[m]a,z)q^{L_0}\Big).
\end{aligned}
$$

If $i\neq -1$, then $\iota_{w,z}(w-z)^i = \iota_{z,w}(w-z)^i = \sum_{n=0}^{i}\binom{i}{n}w^n(-z)^{i-n}$, and the

ith term of the sum is:

$$c_i z^{d_a+d_b-i-1}\Big(\mathrm{Res}_w \sum_{n=0}^{i}\binom{i}{n} w^n(-z)^{i-n}w^{-d_b}\sum_{m=0}^{\infty}\frac{1}{m!}(-1)^m \sum_{r\in\mathbb{Z}+\frac{1}{2}} w^r z^{-r}\frac{r^m q^r}{1+q^r}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$+\mathrm{Res}_w \sum_{n=0}^{i}\binom{i}{n} w^n(-z)^{i-n}w^{-d_b}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r\in\mathbb{Z}+\frac{1}{2}} w^{-r} z^{r}\frac{r^m q^r}{1+q^r}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}\Big)$$

$$= c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=d_b-i-1}^{d_b-1}\binom{i}{d_b-r-1} z^{d_a}(-1)^{i-d_b+r+1}(-1)^m\frac{r^m q^r}{1+q^r}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$+c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-d_b+i+1}\binom{i}{d_b+r-1} z^{d_a}(-1)^{i-d_b-r+1}\frac{r^m q^r}{1+q^r}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$= c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-d_b+i+1}\binom{i}{d_b+r-1} z^{d_a}(-1)^{i-d_b-r+1+m}\frac{(-r)^m q^{-r}}{1+q^{-r}}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$+c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-d_b+i+1}\binom{i}{d_b+r-1} z^{d_a}(-1)^{i-d_b-r+1}\frac{r^m q^r}{1+q^r}\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$= c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-d_b+i+1}\binom{i}{d_b+r-1} z^{d_a}(-1)^{i-d_b-r+1}r^m\Big(\frac{q^{-r}}{1+q^{-r}}+\frac{q^r}{1+q^r}\Big)\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

$$= c_i\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-d_b+i+1}\binom{i}{d_b+r-1} z^{d_a}(-1)^{i-d_b-r+1}r^m\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0}$$

Summing over all i not equal to -1, we have:

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i=0}^{\infty}\sum_{r=-d_b+1}^{-d_b+i+1}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1} r^m\,\mathrm{tr}\,|_M Y(b[m]a,z)q^{L_0} \quad \text{(III.13)}$$

For fixed m the inner double series is not absolutely convergent. However, the subseries of it in which r sums only through negative values *is* absolutely convergent, and so we may switch the order of the inner summations, reindexing appropriately. Then III.13 is rewritten:

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i=d_b-\frac{1}{2}}^{\infty}\sum_{r=\frac{1}{2}}^{-d_b+i+1}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1}r^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}+$$

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=-d_b+1}^{-\frac{1}{2}}\sum_{i=d_b-1+r}^{\infty}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1}r^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}$$

$$=z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i=d_b-\frac{1}{2}}^{\infty}\sum_{r=\frac{1}{2}}^{-d_b+i+1}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1}r^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}+$$

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=\frac{1}{2}}^{d_b-1}\sum_{i=d_b-r-1}^{\infty}(-1)^{i-d_b+r+1}c_i\binom{i}{d_b-r-1}(-r)^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}$$

An elementary examination of the generating series for the c_i reveals that for $k \leq d_b - 1$, $\sum_{i=k}^{\infty}(-1)^{i-k}c_i\binom{i}{k} = 1$. So III.13 reduces to:

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i=d_b-\frac{1}{2}}^{\infty}\sum_{r=\frac{1}{2}}^{-d_b+i+1}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1}r^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}+$$

$$z^{d_a}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{r=\frac{1}{2}}^{d_b-1}(-r)^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}$$

And so the sum in Eq. (III.12) is:

$$z^{d_a}\sum_{m=0}^{\infty}\Big(-2\sum_{r=d_b}{}'\frac{r^{2m+1}q^r}{1+q^r}\frac{1}{(2m+1)!}\operatorname{tr}|_MY(b[2m+1]a,z)q^{L_0}-$$

$$\sum_{r=-d_b+1}^{d_b-1}\frac{r^mq^r}{1+q^r}\frac{1}{m!}\operatorname{tr}|_MY(b[m]a,z)q^{L_0}+\sum_{r=\frac{1}{2}}^{d_b-1}\frac{1}{m!}(-r)^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}+$$

$$\sum_{i=d_b-\frac{1}{2}}^{\infty}\sum_{r=\frac{1}{2}}^{-d_b+i+1}(-1)^{i-d_b-r+1}c_i\binom{i}{d_b+r-1}\frac{1}{m!}r^m\operatorname{tr}|_MY(b[m]a,z)q^{L_0}\Big)$$

$$= z^{d_a} \sum_{m=0}^{\infty} \Big(-2 \sum_{r=d_b}^{\prime} \frac{r^{2m+1} q^r}{1+q^r} \frac{1}{(2m+1)!} \operatorname{tr}|_M Y(b[2m+1]a, z) q^{L_0} -$$

$$2 \sum_{r=\frac{1}{2}}^{d_b-1} \frac{r^{2m+1} q^r}{1+q^r} \frac{1}{(2m+1)!} \operatorname{tr}|_M Y(b[2m+1]a, z) q^{L_0} +$$

$$\sum_{i=d_b-\frac{1}{2}}^{\infty} \sum_{r=\frac{1}{2}}^{-d_b+i+1} (-1)^{i-d_b-r+1} c_i \binom{i}{d_b+r-1} \frac{1}{m!} r^m \operatorname{tr}|_M Y(b[m]a, z) q^{L_0} \Big)$$

$$= z^{d_a} \sum_{m=0}^{\infty} \Big(-2 \sum_{r=\frac{1}{2}}^{\prime} \frac{r^{2m+1} q^r}{1+q^r} \frac{1}{(2m+1)!} \operatorname{tr}|_M Y(b[2m+1]a, z) q^{L_0} +$$

$$\sum_{i=d_b-\frac{1}{2}}^{\infty} \sum_{r=\frac{1}{2}}^{-d_b+i+1} (-1)^{i-d_b-r+1} c_i \binom{i}{d_b+r-1} \frac{1}{m!} r^m \operatorname{tr}|_M Y(b[m]a, z) q^{L_0} \Big)$$

This completes the proof of the theorem. Since $(L_{[-1]}b)[-n] = nb[-n-1]$, we have: □

Corollary III.2. *Let a and b be two odd vectors in the vertex super algebra V, and let M be any V-module. Then*

$$\operatorname{tr}|_M \, o\,(b[-2]a)\, q^{L_0} = -\sum_{m=1}^{\infty} m Q_m^+(q) \operatorname{tr}|_M \, o\,(b[m-1]a)\, q^{L_0}.$$

In the chapters that follow, it is this corollary that will be more often referred to.

CHAPTER IV

THE MAIN THEOREM APPLIED TO NS(5, 3)

An application of the main result of the previous chapter to the Neveu-Schwarz algebras of a certain type yields interesting combinatorial identities. In the course of studying these examples, we discover the values for some of the numbers B_{m,d_b}. (So far, we know them only as series). In this chapter, we specifically examine NS(5, 3). In the following, we examine NS(p, q).

IV.1 The Application to NS(5, 3)

The Verma module for NS with $c_{p,q} = c_{5,3} = \frac{7}{10}$ has a unique (up to scalar) singular vector v_{sing}, and the corresponding quotient yields NS(5, 3). In the case of NS(5, 3),

$$v_{sing} = \left(L_{-2}^2 + \frac{3}{10} L_{-4} - \frac{3}{2} G_{-5/2} G{-3/2} \right) \mathbf{1}.$$

Since the vertex algebra with state-field correspondence $Y(\ , \ )$ is isomorphic to that with $Y[\ , \ ]$, we know that in NS(5, 3),

$$0 = \left(L_{[-2]}^2 + \frac{3}{10} L_{[-4]} - \frac{3}{2} G_{[-5/2]} G_{[-3/2]} \right) \mathbf{1}.$$

Since $2L_{[-4]}\mathbf{1} = L_{[-1]}L_{[-3]}\mathbf{1} = \tilde{\omega}[0]L_{[-3]}\mathbf{1}$, Zhu's theorem tells us that $L_{[-4]}\mathbf{1}$ does not contribute to trace. So we have that:

$$0 = \operatorname{tr}|_M \operatorname{o}\left(L_{[-2]}^2\mathbf{1} - \frac{3}{2}G_{[-5/2]}G_{[-3/2]}\mathbf{1}\right) q^{L_0}.$$

Now $L_{[-2]}L_{[-2]}\mathbf{1} = \tilde{\omega}[-1]\tilde{\omega}$, and $G_{[-5/2]}G_{[-3/2]}\mathbf{1} = \tau[-2]\tau$. So we can apply both Zhu's theorem and the corollary to the main theorem to obtain:

$$\begin{aligned} 0 = \operatorname{tr}|_M \operatorname{o}(\tilde{\omega})\operatorname{o}(\tilde{\omega})q^{L_0} + \sum_{m=1}^{\infty} \tilde{G}_{2m}(q) \operatorname{tr}|_M \operatorname{o}(\tilde{\omega}[2m-1]\tilde{\omega})\, q^{L_0} \\ + \frac{3}{2}\sum_{m=1}^{\infty} \left(mQ_m^+(q) \operatorname{tr}|_M \operatorname{o}(\tau[m-1]\tau)\, q^{L_0}\right). \end{aligned}$$

Given the filtered degrees of $\tilde{\omega}$ and τ (2 and 3/2) and the fact that only nonnegative products appear in this expression, only a few m yield nonzero terms. In fact we only need know the mth product of $\tilde{\omega}$ with itself for $m = 1$ and $m = 3$, and of τ with itself for $m = 0$, $m = 1$, and $m = 2$. This information can be computed by hand, and is summarized:

$$\begin{aligned} \tilde{\omega}[1]\tilde{\omega} &= 2\omega - \frac{7}{120}\mathbf{1} \\ \tilde{\omega}[3]\tilde{\omega} &= \frac{7}{20}\mathbf{1} \\ \tau[0]\tau &= 2\omega - \frac{7}{120}\mathbf{1} \\ \tau[1]\tau &= 0 \\ \tau[2]\tau &= \frac{7}{15}\mathbf{1} \end{aligned}$$

And so the trace equation reduces to:

$$0 = \operatorname{tr}|_M \left(L_0 - \frac{7}{240}\right)^2 q^{L_0} + \tilde{G}_2(q)\operatorname{tr}|_M \left(2L_0 - \frac{7}{120}\right) q^{L_0} + \tilde{G}_4(q)\operatorname{tr}|_M \frac{7}{20} q^{L_0}$$
$$+\frac{3}{2}Q_1^+(q)\operatorname{tr}|_M \left(2L_0 - \frac{7}{120}\right) q^{L_0} + \frac{9}{2}Q_3^+(q)\operatorname{tr}|_M \frac{7}{15} q^{L_0}.$$

Note the presence of the operator $L_0 - \frac{7}{240} = l_0 - \frac{c}{24} = o(\tilde{\omega})$. It suggests we change our degree operator to $\overline{L_0} = L_0 - \frac{c}{24}$. Upon multiplication by $q^{-c/24}$, the equation then reads:

$$0 = \operatorname{tr}|_M \overline{L_0}^2 q^{\overline{L_0}} + 2\tilde{G}_2(q)\operatorname{tr}|_M \overline{L_0} q^{\overline{L_0}} + \tilde{G}_4(q)\operatorname{tr}|_M \frac{7}{20} q^{\overline{L_0}}$$
$$+3Q_1^+(q)\operatorname{tr}|_M \overline{L_0} q^{\overline{L_0}} + \frac{9}{2}Q_3^+(q)\operatorname{tr}|_M \frac{7}{15} q^{\overline{L_0}}.$$

And so the $\overline{L_0}$-character of M must be annihilated by the differential operator

$$D = (q\partial_q)^2 + \left(2\tilde{G}_2(q) + 3Q_1^+(q)\right)(q\partial_q) + \left(\frac{7}{20}\tilde{G}_4(q) + \frac{21}{10}Q_3^+(q)\right).$$

NS(5, 3) has two simple modules, whose characters are known. Their energies are 0 and $\frac{1}{10}$. This implies that their characters have lowest terms q^0 and $q^{1/10}$. So their shifted characters have lowest terms $q^{-7/240}$ and $q^{17/240}$. In light of the above, this means:

$$\left(\frac{-7}{240}\right)^2 + \left(-\frac{1}{6} + 3B_{1.3/2}\right)\left(-\frac{7}{240}\right) + \left(\frac{7}{20}\frac{1}{720} + \frac{7}{20}B_{3,3/2}\right) = 0$$
$$\left(\frac{17}{240}\right)^2 + \left(-\frac{1}{6} + 3B_{1.3/2}\right)\left(\frac{17}{240}\right) + \left(\frac{7}{20}\frac{1}{720} + \frac{7}{20}B_{3,3/2}\right) = 0$$

This allows for the solving of $B_{1,3/2}$ and $B_{3,3/2}$:

$$B_{1,3/2} = \frac{1}{24} \quad \text{(IV.1)}$$

$$B_{3,3/2} = -\frac{7}{960} \quad \text{(IV.2)}$$

Finding these numbers is the important part of the example. We continue studying it to arrive at an interesting q-series identity. The shifted characters of $\mathrm{NS}(5,3)^0$ and $\mathrm{NS}(5,3)^{1/10}$ are determined in [11]:

$$\overline{ch}_0 = q^{-7/240}\eta_0(q)\sum_{k\in\mathbb{Z}}\left(q^{\frac{1}{2}(15k^2+2k)} - q^{\frac{1}{2}(15k^2+8k+1)}\right)$$

$$\overline{ch}_{1/10} = q^{17/240}\eta_0(q)\sum_{k\in\mathbb{Z}}\left(q^{\frac{1}{2}(15k^2-4k)} - q^{\frac{1}{2}(15k^2+14k+3)}\right)$$

An application of the quintuple product identity (see [1]) allows us to rewrite these as:

$$\overline{ch}_0 = q^{-7/240}\eta_0(q)\prod_{n=1}^{\infty}\left(1-q^{5n}\right)\left(1-q^{5n-9/2}\right)\left(1-q^{5n-1/2}\right)\left(1-q^{10n-4}\right)\left(1-q^{10n-6}\right)$$

$$\overline{ch}_{1/10} = q^{17/240}\eta_0(q)\prod_{n=1}^{\infty}\left(1-q^{5n}\right)\left(1-q^{5n-7/2}\right)\left(1-q^{5n-3/2}\right)\left(1-q^{10n-2}\right)\left(1-q^{10n-8}\right)$$

As infinite products, their logarithmic derivatives are easy to establish:

$$\frac{\overline{ch}_0'}{\overline{ch}_0} = -\frac{7}{240}q^{-1} + \sum_{n=1}^{\infty}\left(\frac{(n-1/2)q^{n-3/2}}{1+q^{n-1/2}} - \frac{5nq^{5n-1}}{1-q^{5n}} - \frac{(5n-9/2)q^{5n-11/2}}{1-q^{5n-9/2}} - \right.$$
$$\left.\frac{(5n-1/2)q^{5n-3/2}}{1-q^{5n-1/2}} - \frac{(10n-4)q^{10n-5}}{1-q^{10n-4}} - \frac{(10n-6)q^{10n-7}}{1-q^{10n-6}} + \frac{nq^{n-1}}{(1-q^n)^3}\right)$$

$$\frac{\overline{ch}_{1/10}'}{\overline{ch}_{1/10}} = \frac{17}{240}q^{-1} + \sum_{n=1}^{\infty}\left(\frac{(n-1/2)q^{n-3/2}}{1+q^{n-1/2}} - \frac{5nq^{5n-1}}{1-q^{5n}} - \frac{(5n-7/2)q^{5n-9/2}}{1-q^{5n-7/2}} - \right.$$
$$\left.\frac{(5n-3/2)q^{5n-5/2}}{1-q^{5n-3/2}} - \frac{(10n-2)q^{10n-3}}{1-q^{10n-2}} - \frac{(10n-8)q^{10n-9}}{1-q^{10n-8}} + \frac{nq^{n-1}}{(1-q^n)^3}\right)$$

As fundamental solutions to the differential equation $Df(q) = 0$, the Wronskian of $\overline{ch}_0$ and $\overline{ch}_{1/10}$ is related to the subleading term of D. We compute the Wronskian (where now $'$ means the derivative with respect to τ, where $q = e^{\tau}$):

$$\begin{vmatrix} \overline{ch}_0 & \overline{ch}_{1/10} \\ \overline{ch}_0' & \overline{ch}_{1/10}' \end{vmatrix} = \overline{ch}_0\overline{ch}_{1/10}\left(\frac{1}{10} + \sum_{n=1}^{\infty}\left(\left(\frac{n}{10}\right)\frac{(n/2)q^{n/2}}{1-q^{n/2}} - \left(\frac{n}{5}\right)\frac{(2n)q^{2n}}{1-q^{2n}}\right)\right)$$

Abel's theorem tells us that for some C dependent upon our choice of antiderivative,

$$\overline{ch}_0\overline{ch}_{1/10}\left(\frac{1}{10} + \sum_{n=1}^{\infty}\left(\left(\frac{n}{10}\right)\frac{(n/2)q^{n/2}}{1-q^{n/2}} - \left(\frac{n}{5}\right)\frac{(2n)q^{2n}}{1-q^{2n}}\right)\right) = Ce^{-\int\left(2\tilde{G}_2(q)+3Q_1^+(q)\right)d\tau}$$

The quantity $2\tilde{G}_2(q) + 3Q_1^+(q)$ has an easy-to-describe antiderivative. Since:

$$\left(2\tilde{G}_2(q) + 3Q_1^+(q)\right)q^{-1} = -\frac{1}{24}q^{-1} + 4\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n} - 6\sum_{n=1}^{\infty}\frac{(n-1/2)q^{n-3/2}}{1+q^{n-1/2}}$$

An antiderivative with respect to q is:

$$-\frac{1}{24}\ln(q) - 4\sum_{n=1}^{\infty}\ln(1-q^n) - 6\sum_{n=1}^{\infty}\ln(1+q^{n-1/2}) = -\ln\left(q^{1/24}\prod_{n=1}^{\infty}(1-q^n)^4(1+q^{n-1/2})^6\right) \tag{IV.3}$$

This is the same as an antiderivative for $2\tilde{G}_2(q) + 3Q_1^+(q)$ with respect to τ. So we have that

$$\overline{ch}_0\overline{ch}_{1/10}\left(\frac{1}{10} + \sum_{n=1}^{\infty}\left(\left(\frac{n}{10}\right)\frac{(n/2)q^{n/2}}{1-q^{n/2}} - \left(\frac{n}{5}\right)\frac{(2n)q^{2n}}{1-q^{2n}}\right)\right)$$
$$= Cq^{1/24}\prod_{n=1}^{\infty}(1-q^n)^4(1+q^{n-1/2})^6$$

Recalling the infinite product expressions for $\overline{ch}_0$ and $\overline{ch}_{1/10}$, we have that

$$\frac{1}{10}+\sum_{n=1}^{\infty}\left(\left(\frac{n}{10}\right)\frac{(n/2)q^{n/2}}{1-q^{n/2}}-\left(\frac{n}{5}\right)\frac{(2n)q^{2n}}{1-q^{2n}}\right)$$
$$=C\prod_{n=1}^{\infty}\frac{(1-q^n)^5(1+q^{n-1/2})^5(1-q^{5n-5/2})(1+q^{5n})}{(1-q^{5n})}.$$

Comparing constant terms, C must be $\frac{1}{10}$. So we normalize, and rewrite the right-hand side in terms of η-functions:

$$1+10\sum_{n=1}^{\infty}\left[\left(\frac{n}{10}\right)\frac{(n/2)q^{n/2}}{1-q^{n/2}}-\left(\frac{n}{5}\right)\frac{(2n)q^{2n}}{1-q^{2n}}\right]=\frac{\eta^2(5\tau)\eta_0^5(\tau)}{\eta^{10}(\tau)\eta_0(5\tau)}. \qquad \text{(IV.4)}$$

CHAPTER V

THE CASE OF NS(p, q)

For p and q odd, we can repeat the example of the previous chapter to a point. For $(p, q) = (6k \pm 1, 3)$, we can go further. Not only does this yield a family of identities like IV.4, but it gives us information about the constants $B_{m,d}$ and information about the form of singular vectors in a vertex superalgebra minimal model.

V.1 A Differential operator

We would normally write an even monomial vector v of degree d in NS(p, q) according to the PBW theorem as $L^{j_0}_{[-n]} G^{\epsilon_0}_{[-(n-1/2)]} L^{j_1}_{[-(n-1)]} \cdots L^{j_k}_{[-2]} G^{\epsilon_k}_{[-3/2]} \mathbf{1}$, with $j_i \geq 0$, $j_0 > 0$ or $\epsilon_0 = 1$, and ϵ_i is either 0 or 1. Here we would have $\sum_{i=0}^{k} \Big(j_i(n-i) + \epsilon_i(n-i-1/2) \Big) = d$. The evenness of v would imply $\epsilon_i = 1$ for an even number of i.

The PBW expression for v is inconvenient in what follows. We would prefer to write v as a linear combination of monomial vectors that begin with $L_{[-1]}$, $L_{[-2]}$, or $G_{[-5/2]}$. Each such monomial would be a zeroth product with $\tilde{\omega}$, a minus first product with $\tilde{\omega}$, or a minus second product with τ. The following lemma allows us to do this.

Lemma V.1. *The vector* $v = L^{j_0}_{[-n]} G^{\epsilon_0}_{[-(n-1/2)]} L^{j_1}_{[-(n-1)]} \cdots L^{j_k}_{[-2]} G^{\epsilon_k}_{[-3/2]} \mathbf{1}$ *can be rewritten as a* $\mathbb{C}$*-linear combination of vectors of the form* $L_{[-1]}u$*,* $L_{[-2]}w$ *and* $G_{[-5/2]}x$ *with* u*,* w*, and* x *PBW monomials. Furthermore, if* $v \neq L^{d}_{[-2]}\mathbf{1}$ *and* $v \neq G_{[-5/2]} L^{d-2}_{[-2]} G_{[-3/1]} \mathbf{1}$*, then the leading terms of all* v *and* x *will be* $L_{[-n]}$ *or* $G_{[-n-1/2]}$ *with* $n > 2$*. In other words,*

$$NS(p,q) = L_{[-1]} NS(p,q) + L_{[-2]} NS(p,q) + G_{[-5/2]} NS(p,q). \tag{V.1}$$

Proof. We prove a slightly stronger claim: If a is a PBW monomial, the vectors $L_{[-n]}a$ and $G_{[-(n-1/2)]}a$ can be rewritten as a $\mathbb{C}$-linear combination of vectors of the form $L_{[-1]}u$, $L_{[-2]}v$ and $G_{[-5/2]}w$ with u, v, and w PBW monomials.

The claim is true for $L_{[-2]}a$ and $G_{[-(3-1/2)]}a$. From here we can induct on n using the identities:

$$\begin{aligned} L_{[-n]} &= \frac{1}{n-2}[L_{[-1]}, L_{[-(n-1)]}] \\ G_{[-(n-1/2)]} &= \frac{1}{n-2}[L_{[-1]}, G_{[-((n-1)-1/2)]}]. \end{aligned}$$

After each application of the above identities, we may need to commute terms to the right to write the tail of the vector as the sum of PBW monomials. If $v \neq L^{d}_{[-2]}\mathbf{1}$ and $v \neq G_{[-5/2]} L^{d-2}_{[-2]} G_{[-3/2]}\mathbf{1}$, then the commutations above had to have been applied at least once. This proves that leading terms of all v and x will be $L_{[-n]}$ or $G_{[-n-1/2]}$ with $n > 2$.

∎

The next lemma guarantees the existence of the differential operator generalizing that which appeared in the previous chapter.

Lemma V.2. *For any vector v,*

$$\operatorname{tr}|_M \mathrm{o}(v) q^{\overline{L_0}} = \operatorname{tr}|_M F_v\left(\overline{L_0}\right) q^{\overline{L_0}}$$

where F_v is a polynomial with coefficients in $\mathbb{C}[\tilde{G}_2, \tilde{G}_4, \ldots, Q_1^+, Q_3^+, \ldots]$.

Proof. The claim is immediately verifiable for v of low degree. For v of higher degree, the claim first reduces to PBW monomials via the additivity of trace. The previous lemma reduces it further to the case $v = L_{[-1]}w$, $L_{[-2]}w$ or $G_{[-5/2]}w$. Zhu's theorem tells us that

$$\operatorname{tr}|_M \mathrm{o}(L_{[-1]}w) q^{\overline{L_0}} = 0.$$

Using Zhu's theorem again, and inducting on the degree of v,

$$\begin{aligned}
\operatorname{tr}|_M \mathrm{o}(L_{[-2]}w) q^{\overline{L_0}} &= \operatorname{tr}|_M \mathrm{o}(\tilde{\omega}[-1]w) q^{\overline{L_0}} \\
&= \operatorname{tr}|_M \mathrm{o}(\tilde{\omega})\, \mathrm{o}(w) q^{\overline{L_0}} + \sum_{k=1} \tilde{G}_{2k}(q) \operatorname{tr}|_M \mathrm{o}(\tilde{\omega}[2k-1]w) q^{\overline{L_0}} \\
&= q\partial_q \operatorname{tr}|_M \mathrm{o}(w) q^{\overline{L_0}} + \sum_{k=1} \tilde{G}_{2k}(q) \operatorname{tr}|_M \mathrm{o}(L_{[2k-2]}w) q^{\overline{L_0}} \\
&= q\partial_q \operatorname{tr}|_M F_w\left(\overline{L_0}\right) q^{\overline{L_0}} + \sum_{k=1} \tilde{G}_{2k}(q) \operatorname{tr}|_M F_{L_{[2k-2]}w}\left(\overline{L_0}\right) q^{\overline{L_0}} \\
&= \operatorname{tr}|_M \overline{L_0} F_w\left(\overline{L_0}\right) q^{\overline{L_0}} + \sum_{k=1} \tilde{G}_{2k}(q) \operatorname{tr}|_M F_{L_{[2k-2]}w}\left(\overline{L_0}\right) q^{\overline{L_0}},
\end{aligned}$$

recrsively defining $F_{L_{[-2]}w}$. Similarly, using the corollary to the main theorem and inducting on the degree of v,

$$\begin{aligned}
\operatorname{tr}|_M \, o(G_{[-5/2]}w)q^{\overline{L_0}} &= \operatorname{tr}|_M \, o(\tau[-2]w)q^{\overline{L_0}} \\
&= -\sum_{m=1}^{\infty} mQ_m^+(q) \operatorname{tr}|_M \, o\left(\tau[m-1]w\right) q^{L_0} \\
&= -\sum_{m=1}^{\infty} mQ_m^+(q) \operatorname{tr}|_M \, o\left(G_{[m-3/2]}]w\right) q^{L_0} \\
&= -\sum_{m=1}^{\infty} mQ_m^+(q) \operatorname{tr}|_M F_{G_{[m-3/2]}w}\left(\overline{L_0}\right) q^{L_0},
\end{aligned}$$

recrsively defining $F_{G_{[-5/2]}w}$. □

The next lemma tells us about the leading and subleading coefficients of F_v.

Lemma V.3. *Let* $v = L_{[-n]}^{j_0} G_{[-(n-1/2)]}^{\epsilon_0} L_{[-(n-1)]}^{j_1} \cdots L_{[-2]}^{j_k} G_{[-3/2]}^{\epsilon_k} \mathbf{1}$ *be an even PBW monomial of degree* $2d$.

- *If* $v = L_{[-2]}^{d}\mathbf{1}$, *then* $F_v(X) = X^d + d(d-1)\tilde{G}_2(q)X^{d-1} + \ldots$.
- *If* $v = G_{[-5/2]}L_{[-2]}^{d-2}G_{[-3/2]}\mathbf{1}$, *then* $F_v(X) = -2Q_1^+(q)X^{d-1} + \ldots$
- *Otherwise,* $\deg F_v \leq d-2$

Proof. The first part of this lemma was proved in [10], but we prove the entire lemma here. We induct on the degree of v. First, let $v = L_{[-2]}^{d}\mathbf{1}$. The claim is true when $d = 1$, with $F_v(X) = X$. Now let $d > 1$, and apply Zhu's theorem in what follows. (Appearances of … indicate terms of low enough degree so that the inductive hypothesis guarantees they do not contribute to the degree d or degree $d-1$ coefficient of F_v.)

$$\begin{aligned}
\operatorname{tr}|_M \operatorname{o}(v) q^{\overline{L_0}} &= \operatorname{tr}|_M \operatorname{o}(\tilde{\omega}[-1] L_{[-2]}^{d-1}\mathbf{1}) q^{\overline{L_0}} \\
&= \operatorname{tr}|_M \operatorname{o}(\tilde{\omega})\operatorname{o}(L_{[-2]}^{d-1}\mathbf{1}) q^{\overline{L_0}} \\
&\quad + \sum_{k=1} \tilde{G}_{2k}(q) \operatorname{tr}|_M \operatorname{o}(L_{[2k-1]} L_{[-2]}^{d-1}\mathbf{1}) q^{\overline{L_0}} \\
&= \operatorname{tr}|_M \operatorname{o}(\tilde{\omega})\operatorname{o}(L_{[-2]}^{d-1}\mathbf{1}) q^{\overline{L_0}} + \tilde{G}_2(q) \operatorname{tr}|_M \operatorname{o}(L_{[1]} L_{[-2]}^{d-1}\mathbf{1}) q^{\overline{L_0}} + \dots \\
&= \operatorname{tr}|_M \overline{L_0}\left(\overline{L_0}^{d-1} + (d-1)(d-2)\tilde{G}_2(q)\overline{L_0}^{d-2}\right) q^{\overline{L_0}} \\
&\quad + (2d-2)\tilde{G}_2(q) \operatorname{tr}|_M \left(\overline{L_0}^{d-1} + (d-1)(d-2)\tilde{G}_2(q)\overline{L_0}^{d-2}\right) q^{\overline{L_0}} + \dots \\
&= \operatorname{tr}|_M \left(\overline{L_0}^{d} + d(d-1)\tilde{G}_2(q)\overline{L_0}^{d-1}\right) q^{\overline{L_0}} + \dots,
\end{aligned}$$

inductively proving the claim for $v = L_{[-2]}^{d}\mathbf{1}$.

Next, if $v = G_{[-5/2]} L_{[-2]}^{d-2} G_{[-3/2]}\mathbf{1}$, then the corollary to main theorem shows

$$\begin{aligned}
\operatorname{tr}|_M \operatorname{o}(v) q^{\overline{L_0}} &= \operatorname{tr}|_M \operatorname{o}(\tau[-2] L_{[-2]}^{d-2} G_{[-3/2]}\mathbf{1}) q^{\overline{L_0}} \\
&= -\sum_{m=1} m Q_m^{+}(q) \operatorname{tr}|_M \operatorname{o}\left(\tau[m-1] L_{[-2]}^{d-2} G_{[-3/2]}\mathbf{1}\right) q^{\overline{L_0}} \\
&= -Q_1^{+}(q) \operatorname{tr}|_M \operatorname{o}\left(G_{[-1/2]} L_{[-2]}^{d-2} G_{[-3/2]}\mathbf{1}\right) q^{\overline{L_0}} + \dots.
\end{aligned}$$

We can commute the $G_{[-1/2]}$ all the way to the right and see inductively that almost all terms arising have degree less than our concern. The exception is when $G_{[-1/2]}$ commutes past $G_{[-3/2]}$, producing $2L_{[-2]}$. So we have

$$\operatorname{tr}|_M \operatorname{o}(v) q^{\overline{L_0}} = -Q_1^{+}(q) \operatorname{tr}|_M \operatorname{o}\left(2L_{[-2]}^{d-1}\mathbf{1}\right) q^{\overline{L_0}} + \dots,$$

inductively proving the claim for $v = G_{[-5/2]} L_{[-2]}^{d-2} G_{[-3/2]}\mathbf{1}$.

Finally, let v be otherwise. That is, let the leading term of v be $L_{[-n]}$ or

$G_{[-n-1/2]}$ with $n > 2$. Then lemma V.1 allows us to rewrite v as a sum of terms of the form $L_{[-1]}u$, $L_{[-1]}w$, and $G_{[-5/2]}x$ in such a way so that the leading terms of all w and x are $L_{[-n]}$ or $G_{[-n-1/2]}$ with $n > 2$. Also, all w and x have smaller degrees than v. Zhu's theorem and the main theorem inductively imply the third claim of the lemma.

□

V.2 Singular Vectors and $NS(p, q)$

The Verma module for NS obtained in with charge $c_{p,q}$ has a unique singular vector v_{sing} of homogeneous degree $2\delta = \frac{(p-1)(q-1)}{2}$ (see [11]). In the case where p and q are both odd, we may normalize to write

$$v_{sing} \quad = \quad L_{-2}^{\delta}\mathbf{1} + \lambda_{p,q} G_{-5/2} L_{-2}^{\delta-2} G_{-3/2}\mathbf{1} + \ldots$$

where $\lambda_{p,q}$ is an as yet undetermined constant. Just as with NS(5, 3), we consider the trace on v_{sing}. We can apply lemmas V.2 and V.3 to obtain

$$\begin{aligned} 0 &= \operatorname{tr}|_M \, o\,(v_{sing})\, q^{\overline{L_0}} \\ &= \operatorname{tr}|_M F_{v_{sing}}\left(\overline{L_0}\right) q^{\overline{L_0}} \\ &= \operatorname{tr}|_M \left(\overline{L_0}^{\delta} + \left[\delta(\delta-1)\tilde{G}_{2k} - 2\lambda_{p,q} Q_1^{+}(q)\right] \overline{L_0}^{\delta-1} + \ldots\right) q^{\overline{L_0}} \end{aligned}$$

And so the $\overline{L_0}$ characters of the δ simple representations of NS(p, q) are annihilated by a differential operator:

$$D = (q\partial_q)^{\delta} + \left[\delta(\delta-1)\tilde{G}_{2k} - 2\lambda_{p,q} Q_1^{+}(q)\right] (q\partial_q)^{\delta-1} + \ldots .$$

The charge and energies of NS(p,q) are given by

$$c = \frac{3}{2}\left(1 - \frac{2(p-q)^2)}{pq}\right) \quad \text{(V.2)}$$

$$h^{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{8pq}, \quad \text{(V.3)}$$

where $1 \leq s < p$, $1 \leq r < q$, and r and s have matching parity. In [11] the characters of the NS$(p,q)^h$ were established to be

$$ch_h = \eta_0(q) \sum_{k\in\mathbb{Z}} \left(q^{b_k} - q^{a_k}\right) \quad \text{(V.4)}$$

where

$$a_k = \frac{(pr+qs+2pqk)^2 - (p-q)^2}{8pq}$$

$$b_k = \frac{(pr-qs+2pqk)^2 - (p-q)^2}{8pq}$$

We will denote the $\overline{L_0}$-character of NS$(p,q)^h$ by $\overline{ch}_h$. The collection of $\overline{ch}_h$ form a fundamental set of solutions to the differential equation $DF(q) = 0$, where D is the differential operator established in the previous section. And so Abel's theorem tells us:

$$\begin{vmatrix} \overline{ch}_{h_1} & \overline{ch}_{h_2} & \cdots & \overline{ch}_{h_\delta} \\ \overline{ch}_{h_1}^{(1)} & \overline{ch}_{h_2}^{(1)} & & \overline{ch}_{h_\delta}^{(1)} \\ \vdots & & \ddots & \vdots \\ \overline{ch}_{h_1}^{(\delta-1)} & \overline{ch}_{h_2}^{(\delta-1)} & \cdots & \overline{ch}_{h_\delta}^{(\delta-1)} \end{vmatrix} = C\, e^{\int\left[\delta(\delta-1)\tilde{G}_{2k} - 2\lambda_{p,q} Q_1^+(q)\right] d\tau} \quad \text{(V.5)}$$

$$= C\, q^{\frac{\delta(\delta-1)+\lambda_{p,q}}{12}} \prod_{n=1}^{\infty} \frac{(1-q^n)^{2\delta(\delta-1)}}{(1+q^{n-1/2})^{4\lambda_{p,q}}}. \quad \text{(V.6)}$$

The leading exponent in $\overline{ch}_{h_i}^{(j)}$ is $h_i - \frac{c}{24}$. So the leading exponent in the Wronskian on the left is $\sum_i (h_i - \frac{c}{24})$. Using the formulas V.3 and V.2, and summing over values of r and s, one can show $\sum_i (h_i - \frac{c}{24}) = \frac{(pq-p-q+1)(pq-p-q-6)}{192}$. Equating the leading exponent on both sides of V.6 yields

$$(pq-p-q+1)(pq-p-q-6) = 16\delta(\delta-1) + 16\lambda_{p,q}.$$

It is not hard to see that $\delta = \frac{(p-1)(q-1)}{4}$. So

$$\begin{aligned}(pq-p-q+1)(pq-p-q-6) &= (p-1)(q-1)(pq-p-q-3) + 16\lambda_{p,q} \\ \implies -3(pq-p-q+1) &= 16\lambda_{p,q} \\ \implies \lambda_{p,q} &= -\frac{3}{16}(p-1)(q-1).\end{aligned}$$

We have proved

Theorem V.1. *For odd p, q, the singular vector of* $\mathrm{NS}(p,q)$ *has first few PBW terms*

$$L_{[-2]}^{\delta}\boldsymbol{1} - \frac{3}{16}(p-1)(q-1)G_{[-5/2]}L_{[-2]}^{\delta-2}G_{[-3/2]}\boldsymbol{1} + \dots.$$

or

$$L_{[-2]}^{\delta}\boldsymbol{1} - \frac{3}{4}\delta G_{[-5/2]}L_{[-2]}^{\delta-2}G_{[-3/2]}\boldsymbol{1} + \dots.$$

We can also compare the leading coefficients of V.6 to determine the value of C. Since $\overline{ch}_{h_i} = q^{h_i - \frac{c}{24}} + \dots$, the leading coeffecient of $\overline{ch}_{h_i}^{(j)}$ is $(h_i - \frac{c}{24})^j$. And so the leading coefficient in equation V.6 is the Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (h_1 - \frac{c}{24})^1 & (h_2 - \frac{c}{24})^1 & & (h_\delta - \frac{c}{24})^1 \\ \vdots & & \ddots & \vdots \\ (h_1 - \frac{c}{24})^{\delta-1} & (h_2 - \frac{c}{24})^{\delta-1} & \cdots & (h_\delta - \frac{c}{24})^{\delta-1} \end{vmatrix} = \prod_{i>j}(h_i - h_j)$$

We have proved

Theorem V.2. *Let $V = \mathrm{NS}(p,q)$ with p, q odd. If h_i denotes the energies of the $\delta = \frac{(p-1)(q-1)}{4}$ simple represntations and $\overline{ch}_{h_i}$ denotes the shifted (by $c/24$) characters, then the Wronskian of the $\overline{ch}_{h_i}$*

$$W(\overline{ch}_{h_i}) = q^{\frac{(pq-p-q+1)(pq-p-q-6)}{192}} \prod_{n=1}^{\infty}(1-q^n)^{2\delta(\delta-1)}(1+q^{n-1/2})^{\frac{3}{4}(p-1)(q-1)} \prod_{i>j}(h_i - h_j).$$

V.3 NS$(6k \pm 1, 3)$

The consequences of theorem V.2 can be followed through a bit further in the special case $q = 3$. In this case all the energies $h^{r,s}$ can be chosen with $r = 1$, $s = 2i - 1$. This in turn allows for an application of the Watson quintuple product ([1]) to $\overline{ch}_{h_i}$. We have

$$\begin{aligned}
\overline{ch}_{h_{1,s}} &= q^{-c/24}\eta_0(q)\sum_{k\in\mathbb{Z}}\left(q^{\frac{(p-qs+2pqk)^2-(p-q)^2}{8pq}} - q^{\frac{(p+qs+2pqk)^2-(p-q)^2}{8pq}}\right)\\
&= q^{-c/24+\frac{(p-qs)^2-(p-q)^2}{8pq}}\eta_0(q)\sum_{k\in\mathbb{Z}}\left(q^{\frac{(p-qs)k+pqk^2}{2}} - q^{\frac{s+(p+q(2i-1))k+pqk^2}{2}}\right)\\
&= q^{-c/24+\frac{(p-qs)^2-(p-q)^2}{8pq}}\eta_0(q)\sum_{k\in\mathbb{Z}}\left(q^{\frac{(p-qs)k+pqk^2}{2}} - q^{\frac{s+(p+qs)k+pqk^2}{2}}\right)\\
&= q^{-c/24+\frac{(p-qs)^2-(p-q)^2}{8pq}}\eta_0(q)\sum_{k\in\mathbb{Z}}\left((q^{\frac{-s}{2}})^{3k} - (q^{\frac{-s}{2}})^{-1-3k}\right)(q^p)^{\frac{k(3k+1)}{2}}\\
&= q^{c_s}\eta_0(q)\prod_{n\geq 1}(1-q^{pn})(1-q^{pn-\frac{s}{2}})(1-q^{p(n-1)+\frac{s}{2}})(1-q^{p(2n-1)-s})(1-q^{p(2n-1)+s}).
\end{aligned}$$

The point is that we can express $\overline{ch}_{h_{1,s}}$ as an infinite product, $\prod_n A_{s,n}$. In that case, $\overline{ch}^{(1)}_{h_{1,s}} = \prod_n A_{s,n}\left(\sum_m A^{(1)}_{s,m}/A_{s,m}\right) = \overline{ch}_{h_{1,s}}\cdot S_s$. Inductively, we can show

$$\overline{ch}^{(i)}_{h_{1,s}} = \overline{ch}_{h_{1,s}}\cdot(\partial_\tau + S_s)^i 1.$$

So theorem V.2 and the above imply

Corollary V.3.

$$\begin{aligned}
&\prod_s \overline{ch}_{h_{1,s}}\begin{vmatrix} (\partial_\tau+S_1)^0 1 & \cdots & (\partial_\tau+S_{p-2})^0 1\\ \vdots & \ddots & \vdots\\ (\partial_\tau+S_1)^{\delta-1}1 & \cdots & (\partial_\tau+S_{p-2})^{\delta-1}1\end{vmatrix}\\
&\quad = \prod_{i>j}(h_{1,i}-h_{1,j})q^{\frac{(2p-2)(2p-9)}{192}}\prod_{n=1}^{\infty}(1-q^n)^{\frac{(p-1)(p-3)}{2}}(1+q^{n-1/2})^{\frac{3}{2}(p-1)}.
\end{aligned}$$

CHAPTER VI

FUTURE INQUIRIES

VI.1 Improving Upon the Differential Operator

Lemma V.3 used induction to determine the leading terms of F_v for two types of PBW monomials. For the purposes of chapter V this was enough; only these two types of PBW monomials in the expression of v_{sing} contribute to the subleading term of the differential operator V.2.

A perfect improvement of lemma V.3 would give a closed formula for F_v whenever v was a PBW monomial. This seems hopeless, as the diversity of PBW monomials grows with the partition function. However it is still within reason to ask for a theorem that expresses F_v as a sum over partitions. This could be useful in determining more precise information about the form of v_{sing}. In NS(p, q) it has been known since [9] that v_{sing} lies in $V_{\frac{(p-1)(q-1)}{2}}$ when p and q are odd, and in $V_{\frac{(p-1)(q-1)+1}{2}}$ when p and q are even. But beyond this and theorem V.1, nothing is universally known about v_{sing}.

VI.2 Quasi-Bernoulli Numbers

In the process of examining NS(5, 3) we determined the values for $B_{1,3/2}$ and $B_{3,3/2}$. Recall that in general,

$$B_{m,d} = \sum_{i=d-1/2}^{\infty} (-1)^{i-d+1/2} c_{d,i} \sum_{n=0}^{i-d+1/2} (-1)^n \binom{i}{d+n-1/2} (n+1/2)^m,$$

where the $c_{d,i}$ are defined by $(1+z)^{d-1}\ln(1+z)^{-1} = \sum c_{d,i} z^i$. These numbers seem to be some kind of generalization of the Bernoulli numbers

$$B_m = \sum_{i=0}^{m} \frac{1}{i+1} \sum_{n=0}^{i} (-1)^n \binom{i}{n} (n)^m.$$

At present this is just an observation, but there is undoubtedly more meaning to it. The series $Q_m^+(q)$ no doubt have modularity properties generalizing those of $\tilde{G}_{2k}(q)$. The constant terms of $\tilde{G}_{2k}(q)$ are Bernoulli numbers (scaled by factorials), and we believe there should be a meaningful way to relate the constant terms of $Q_m^+(q)$ (the $B_{m,d}$) to the Bernoulli numbers.

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