On bounded dominance criteria*

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Abstract

A well-known criterion to make heterogeneous welfare comparisons is Atkinson and Bourguignon's (1987) sequential generalized Lorenz dominance (SGLD) criterion. Recently, Fleurbaey, Hagneré and Trannoy (2003) convincingly argue that it contains unreasonable household utility profiles and suggest to put (lower and upper) bounds on the needs of the different household types. *First*, we generalize Atkinson and Bourguignon's SGLD criterion, by introducing lower bounds in the household utility profiles. *Second*, we propose a new SGLD criterion by introducing upper bounds in a similar way. *Third*, we impose lower and upper bounds simultaneously and obtain a criterion which is intermediate between Ebert's (1999) equivalence scale weighted approach and Atkinson and Bourguignon's (1987) SGLD approach.

1 Introduction

If we do not want to cardinalize needs differences via equivalence scales (Ebert, 1997, 1999, Shorrocks, 2004), the most well-known way to make heterogeneous welfare comparisons is the so-called "ordinal" sequential generalized Lorenz dominance (SGLD) test. It boils down to classifying households in different needs groups and checking —on the basis of the generalized Lorenz dominance criterion applied to household incomes— whether the most needy are better off, whether the most and second most needy are better off, and so on. This result is due to Atkinson and Bourguignon (1987), and extended by Atkinson (1992), Jenkins and Lambert (1993), Chambaz and Maurin (1998), Moyes (1999) and Lambert and Ramos (2002) to deal with changing demographics, poverty and/or the principle of diminishing transfers.

As noted, none of these results are predicated upon equivalence scales. A recent suggestion of Fleurbaey, Hagneré and Trannoy (2003), building on work of Bourguignon

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(1989), allows for the use of a variety of equivalence scales, not specified except for lying between lower and upper bounds. Motivated by this idea, we here first generalize Atkinson and Bourguignon's SGLD criterion by introducing lower bounds in the household utility profiles. Such a lower bound tells us, for example, that a couple with an income equal to or lower than αy (with $\alpha \geq 1$) is worse off than a single with income y (for all income levels y). In section 3, we describe the corresponding sequential procedure applied to household incomes, which have to be divided and weighted[1] by (a multiplication of) the lower bounds; if all lower bounds equal 1, we are back in the standard case.

Second, in an analogous way, we introduce upper bounds in the household utility profiles. Such an upper bound tells us, for example, that a couple with an income equal to or higher than βy (with $\beta \geq \alpha$) is better off than a single with income y (for all income levels y). In section 4, we introduce a *reversed* sequential procedure applied to household incomes, which again have to be divided and weighted in this case by (a multiplication of) the upper bounds, but the sequencing is exactly the opposite. This stands to reason: if we divide the household income of the couple by the upper bound β, singles are more needy than couples whenever they have the same scaled income. One specific case deserves further attention. Choosing upper bounds on the basis of household size, we obtain a sequential dominance procedure applied to distributions of *individuals* with *per-capita* incomes, but, in contrast with Jenkins and Lambert's (1993, p. 343) proposal, starting from the singles, followed by singles and couples, and so on.

Third, in the spirit of Fleurbaey et al. (2003), we consider lower and upper bounds simultaneously in Atkinson and Bourguignon's setting. We obtain in section 5 an implementable criterion, which reduces to Ebert's (1999) approach when the lower bounds equal the upper bounds and to (Jenkins and Lambert's extension of) Atkinson and Bourguignon's (1987) SGLD criterion when the lower bounds equal one and the upper bounds approach infinity.

2 Notation

Consider household incomes $y \in \mathbb{R}_+$ and household types $k \in \mathbb{K} = \{1, ..., K\}$. Types are ordered from least to most needy (given the same household income); as such, k could be household size. The well-being level of a type k household as a function of income is measured via a (twice continuously differentiable) household utility function $U_k : \mathbb{R}_+ \to \mathbb{R}$. A heterogeneous distribution consists of (i) proportions of type k households, denoted p_k, with $\sum_{k \in \mathbb{K}} p_k = 1$, and (ii) income distribution functions of type k households, denoted F_k, assumed to be continuously differentiable and defined

[1] The weighting procedure is the same as the one proposed in Ebert (1997, 1999) and Ebert and Moyes (2003).

over a finite support $[\underline{s}_k, \overline{s}_k]$ (and thus equal to zero or one outside this support), with all $\overline{s}_k > \underline{s}_k > 0$. We abbreviate a distribution as $F = (p_1, \ldots, p_K, F_1, \ldots, F_K)$ and $G = (q_1, \ldots, q_K, G_1, \ldots, G_K)$ denotes an alternative distribution. For brevity, we thus directly focus on the case where demographics might be different between distributions. We want to derive an implementable criterion which tells us whether the difference in average utility between two distributions

$$\Delta W_U = \sum_{k \in \mathbb{K}} \int_{\underline{s}_k}^{\overline{s}_k} U_k(y)\, d\left(p_k F_k(y) - q_k G_k(y)\right), \tag{1}$$

is positive (or negative) for all utility profiles $U = (U_1, \ldots, U_K)$ satisfying certain properties. The reasonableness of such a criterion clearly depends upon the reasonableness of the properties we impose on the utility profiles. In the sequel, we always focus on utility profiles $U = (U_1, \ldots, U_K)$ where the marginal utility of income[2] —called *social priority*— of all household types is positive (A1a), but decreases with income (A1b):

A1: $U_k' \geq 0$, for all $k \in \mathbb{K}$ (A1a) and $U_k'' \leq 0$, for all $k \in \mathbb{K}$ (A1b).

In terms of income transfers, A1a —known as the Pareto condition— ensures that more income for any household improves social welfare, whereas A1b —known as the (within type) Pigou-Dalton transfer principle— tells us that an income transfer from a richer to a poorer household of the same type increases welfare.

3 Lower bounds

A lower bound vector is defined as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K) \geq (1, \ldots, 1)$; we choose the least needy type, type 1, as the reference type, or $\alpha_1 = 1$. As in Fleurbaey et al. (2003), these lower bounds capture judgements about the needs differences between adjacent types only; for later use, we define an equivalence scale vector with respect to the reference type, as $\boldsymbol{\alpha}^* = (\alpha_1^*, \ldots, \alpha_K^*)$ with $\alpha_k^* = \prod_{i=1}^k \alpha_i$. We might impose the following conditions on utility profiles $(U_1, \ldots, U_K)$ for a given lower bound vector $\boldsymbol{\alpha}$ and a maximum income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \frac{\overline{s}_2}{\alpha_2^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$ (an explanation follows):

$\mathbf{A2}_{\boldsymbol{\alpha}}$: $U_k'(\alpha_k y) - U_{k-1}'(y) \geq 0$, for all $y \in \mathbb{R}_+$, for all $k = 2, \ldots, K$.

$\mathbf{A3}_{\boldsymbol{\alpha}}$: $\left(U_k'(\alpha_k y) - U_{k-1}'(y)\right)' \leq 0$, for all $y \in \mathbb{R}_+$, for all $k = 2, \ldots, K$.

$\mathbf{A4}_{\boldsymbol{\alpha}}$: $U_k(\alpha_k^* \overline{a}) = U_{k-1}(\alpha_{k-1}^* \overline{a})$, for all $k = 2, \ldots, K$.

[2] It tells us where to put our money first as a social planner who maximizes the average household utility.

Assumption A2$_{\boldsymbol{\alpha}}$ is due to Fleurbaey et al. (2003): together with A1, it tells us that a household of type k has a higher social priority compared to a household of type $k-1$, if the former's household income is sufficiently low, i.e., lower than α_k times the latter's household income. Assumption A3$_{\boldsymbol{\alpha}}$ is an adaptation of Atkinson and Bourguignon's condition: it tells us that the previous difference in social priority (described in A2$_{\boldsymbol{\alpha}}$) decreases with income. Assumption A4$_{\boldsymbol{\alpha}}$ is an adaptation of Jenkins and Lambert's (1993) condition to deal with changing demographics: there exists incomes $\alpha_k^* \overline{a}$ where utility levels become equal. Somewhat weaker, one could use a variant of Moyes' (1999) condition, which, together with A1, says that a household of type k is worse off than a household of type $k-1$, if the former's household income is sufficiently low, i.e., lower than α_k times the latter's household income:

$$U_k(\alpha_k^* y) \leq U_{k-1}(\alpha_{k-1}^* y), \text{ for all } y \in \mathbb{R}_+, \text{ for all } k = 2, \ldots, K.$$

This would add the conditions $\sum_{i=k}^{K} (p_i - q_i) \leq 0$ for all $k \in \mathbb{K}$ to proposition 1 below. Choosing $\boldsymbol{\alpha} = \mathbf{1} = (1, \ldots, 1)$, the above assumptions reduce to the ones considered by Bourguignon (A1, A2$_{\mathbf{1}}$), Atkinson and Bourguignon (A1, A2$_{\mathbf{1}}$, A3$_{\mathbf{1}}$) and Jenkins and Lambert and Chambaz and Maurin (A1, A2$_{\mathbf{1}}$, A3$_{\mathbf{1}}$, A4$_{\mathbf{1}}$).

We denote with $\mathcal{U}_{\boldsymbol{\alpha}}$ the family of utility profiles $U = (U_1, ..., U_K)$ satisfying assumptions A1, A2, A3$_{\boldsymbol{\alpha}}$, and A4$_{\boldsymbol{\alpha}}$, for a given $\boldsymbol{\alpha}$ (and given $\overline{a}$, which will remain fixed, and may be omitted in the sequel). We say that a distribution F welfare dominates G according to the family $\mathcal{U}_{\boldsymbol{\alpha}}$, denoted $F \succsim_{\boldsymbol{\alpha}} G$, if and only if the welfare difference ΔW_U, defined in (1), is non-negative for all profiles U in $\mathcal{U}_{\boldsymbol{\alpha}}$. The following proposition shows how the welfare dominance quasi-ordering $\succsim_{\boldsymbol{\alpha}}$ can be implemented via sequential dominance conditions (a discussion follows):[3]

Proposition 1. *Consider two heterogeneous distributions F and G as well as a lower bound vector $\boldsymbol{\alpha} \in \mathbb{R}^K$ with $\alpha_k \geq \alpha_1 = 1$ for all $k \in \mathbb{K}$ and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$, with $\alpha_k^* = \prod_{i=1}^{k} \alpha_i$ for all $k \in \mathbb{K}$. We have*

$$F \succsim_{\boldsymbol{\alpha}} G \Leftrightarrow \sum_{i=k}^{K} H_i\left(\alpha_i^* y\right) \leq 0, \textit{ for all } y \in [0, \overline{a}] \textit{ and for all } k \in \mathbb{K}, \tag{2}$$

with $H_k : \mathbb{R}_+ \to \mathbb{R} : y \mapsto \int_0^y \left(p_k F_k\left(x\right) - q_k G_k\left(x\right)\right) dx$, for all $k \in \mathbb{K}$.

The sequential conditions are applied to household incomes —which are both divided and weighted by the lower bound equivalence scales in $\boldsymbol{\alpha}^*$, as in Ebert (1997,1999)— and starting from the most needy type, followed by the most and second most needy types, and so on. Choosing $\boldsymbol{\alpha} = \mathbf{1}$, our criterion reduces to Jenkins and Lambert's

[3] All proofs can be found in the appendix.

extension of the SGLD criterion. If, in addition, demographics are the same in both distributions, then $\succsim_{\boldsymbol{\alpha}}$ is equivalent with Atkinson and Bourguignon's (1987) SGLD criterion.

4 Upper bounds

In a similar way, we might also impose upper bounds via a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_K) \geq (1, \ldots, 1)$, reflecting the idea that there are limits to the needs of the "more needy". Again type 1 is the reference type, or $\beta_1 = 1$, and we define an equivalence scale $\beta_k^* = \prod_{i=1}^k \beta_i$ with respect to this reference type. For example, it is generally accepted that a couple does not need more than twice the income of a single to reach the same living standards. We might impose one of the following conditions on utility profiles $(U_1, \ldots, U_K)$ for a given upper bound vector $\boldsymbol{\beta}$ and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\beta_1^*}, \frac{\overline{s}_2}{\beta_2^*}, \ldots, \frac{\overline{s}_K}{\beta_K^*}\right)$:

$\mathbf{A2}^{\boldsymbol{\beta}}: U_k'(\beta_k y) - U_{k-1}'(y) \leq 0$, for all $y \in \mathbb{R}_+$, for all $k = 2, \ldots, K$.

$\mathbf{A3}^{\boldsymbol{\beta}}: \left(U_k'(\beta_k y) - U_{k-1}'(y)\right)' \geq 0$, for all $y \in \mathbb{R}_+$, for all $k = 2, \ldots, K$.

$\mathbf{A4}^{\boldsymbol{\beta}}: U_k\left(\beta_k^* \overline{a}\right) = U_{k-1}\left(\beta_{k-1}^* \overline{a}\right)$, for all $k = 2, \ldots, K$.

The interpretation is much as before. Assumption A2$^{\boldsymbol{\beta}}$ is again due to Fleurbaey et al. (2003): together with A1, it tells us that a household with type k has a lower social priority compared to a household with type $k-1$, if the former's household income is sufficiently high, i.e., higher than β_k times the latter's household income. Assumption A3$^{\boldsymbol{\beta}}$ tells us that the (positive) difference in social priority, described in assumption A2$^{\boldsymbol{\beta}}$, becomes less important when incomes grow larger. Finally, according to assumption A4$^{\boldsymbol{\beta}}$ there exist incomes $\beta_k^* \overline{a}$ where all utility levels become equal.[4]

Let $\succsim^{\boldsymbol{\beta}}$ be the quasi-ordering which corresponds with welfare dominance according to all profiles in the family $\mathcal{U}^{\boldsymbol{\beta}}$, defined as the family of utility profiles $(U_1, ..., U_K)$ satisfying assumptions A1, A2$^{\boldsymbol{\beta}}$, A3$^{\boldsymbol{\beta}}$, and A4$^{\boldsymbol{\beta}}$, for a given upper bound vector $\boldsymbol{\beta}$ (and given $\overline{a}$). Our next proposition shows how welfare dominance for $\succsim^{\boldsymbol{\beta}}$ can be implemented via sequential dominance conditions:

Proposition 2. *Consider two heterogeneous distributions F and G as well as an upper bound vector $\boldsymbol{\beta} \in \mathbb{R}^K$ with $\beta_k \geq \beta_1 = 1$ for all $k \in \mathbb{K}$ and an exogeneous income level*

[4] Also here, one could use a variant of Moyes' (1999) condition, $U_k(\beta_k^* y) \geq U_{k-1}(\beta_{k-1}^* y)$, for all $y \in \mathbb{R}_+$, for all $k = 2, \ldots, K$, (with a similar interpretation as before) which would add the condition $\sum_{i=1}^k (p_i - q_i) \leq 0$ for all $k \in \mathbb{K}$ to proposition 2.

$\overline{a} \geq \max\left(\frac{\overline{s}_1}{\beta_1^*}, \ldots, \frac{\overline{s}_K}{\beta_K^*}\right)$, *with* $\beta_k^* = \prod_{i=1}^k \beta_i$ *for all* $k \in \mathbb{K}$. *We have*

$$F \succsim^{\boldsymbol{\beta}} G \Leftrightarrow \sum_{i=1}^{k} H_i\left(\beta_i^* y\right) \leq 0, \text{ for all } y \in [0, \overline{a}] \text{ and for all } k \in \mathbb{K}, \tag{3}$$

with all functions H_k *defined as in proposition 1.*

The sequential conditions are again applied to household incomes —which are both divided and weighted, here by the upper bound equivalence scales $\boldsymbol{\beta}^*$, as in Ebert (1997,1999)— but starting from the least needy type, followed by the least and second least needy types, and so on. If k equals household size, we could choose $\beta_k = \frac{k}{k-1}$, for all $k = 2, \ldots, K$, and thus $\beta_k^* = k$ as upper bounds, expressing the view that there are economies of scale in household size. In this specific case, the above criterion reduces to a sequential dominance criterion applied to the per-capita incomes of individuals, starting from singles only, singles and couples together, and so on. The sequence is indeed reversed, because, for the same per-capita income, singles are most needy, followed by couples and so on. These conditions have not to our knowledge been seen before in the welfare dominance literature.

5 Lower and upper bounds

The lower bound criterion in section 3 can deal with (i) transfers from richer households to poorer and more needy households —where richer and poorer have to be understood here in terms of equivalent incomes— but not with (ii) transfers from richer households to poorer and *less* needy households; exactly the opposite holds true for the upper bound criterion described in section 4. It is therefore tempting to introduce lower and upper bounds simultaneously as in Fleurbaey et al. (2003), who consider assumptions A1, A2$_{\boldsymbol{\alpha}}$, A2$^{\boldsymbol{\beta}}$ for some lower and upper bound vectors $\boldsymbol{\beta} \geq \boldsymbol{\alpha} \geq \mathbf{1}$. Let $\succsim_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ be the quasi-ordering which corresponds with welfare dominance according to all profiles in the family $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} = \mathcal{U}_{\boldsymbol{\alpha}} \cap \mathcal{U}^{\boldsymbol{\beta}}$, i.e., the family of utility profiles $(U_1, ..., U_K)$ satisfying assumptions A1, A2$_{\boldsymbol{\alpha}}$, A2$^{\boldsymbol{\beta}}$, A3$_{\boldsymbol{\alpha}}$, A3$^{\boldsymbol{\beta}}$, A4$_{\boldsymbol{\alpha}}$ and A4$^{\boldsymbol{\beta}}$, given upper and lower bound vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ (and given $\overline{a}$) which satisfy $\boldsymbol{\beta} \geq \boldsymbol{\alpha} \geq \mathbf{1}$.

Note first of all that if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\alpha_k^* = \beta_k^*$ ($= m_k$, say) is an agreed equivalence scale, then utility profiles $(U_1, ..., U_K)$ for which $U_k(y) = m_k U_1\left(\frac{y}{m_k}\right)$ (for all $y \in \mathbb{R}_+$ and for all $k \in \mathbb{K}$) where U_1 is increasing and concave, not only reconcile the Atkinson-Bourguignon and Ebert approaches (indeed are obligatory if the normative approach of Ebert and Moyes (2003) is endorsed), but also belong to $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ (for $\boldsymbol{\alpha} = \boldsymbol{\beta}$), although in this trivial case of an agreed equivalence scale nothing new emerges, of course.

Assuming that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, and setting $y = 0$ in A2$_{\boldsymbol{\alpha}}$ and A2$_{\boldsymbol{\beta}}$, $U_1'(0) = U_2'(0) = \cdots = U_K'(0)$ is implied of any utility profile $(U_1, ..., U_K)$ belonging to $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$. If this common marginal utility value at the origin is finite, then from A3$_{\boldsymbol{\alpha}}$ and A3$^{\boldsymbol{\beta}}$, there exist scalars $b \geq 0$ and a_k, for all $k \in \mathbb{K}$, such that $U_k(y) = by + a_k$ (for all $y \in \mathbb{R}_+$ and for all $k \in \mathbb{K}$). In order that A4$_{\boldsymbol{\alpha}}$ and A4$^{\boldsymbol{\beta}}$ should also hold, along with twice differentiability of all utility functions, $b = 0$ is then required. Hence, if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, all *non-trivial* profiles of utility functions in $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ have infinite first derivatives at $y = 0$. In fact, for each utility profile in $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$, conditions

$$U_k(y) = \overline{U} \text{ and } U_k^{'}(y) = \overline{U}^{'} \text{ for all } \alpha_k^* \overline{a} \leq y \leq \beta_k^* \overline{a} \text{ and for all } k \in \mathbb{K}$$

are implied for appropriate scalars $\overline{U}$ and $\overline{U}^{'}$, the latter of which is of course zero if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ (and then $U_k(y) = \overline{U}$ for all $y \geq \alpha_k^* \overline{a}$ and for all $k \in \mathbb{K}$). Fleurbaey et al.'s (2003) condition to deal with changing demographics is weaker. Given $\overline{a}$, it boils down to

A5 : There exists a vector $(a_2, \ldots, a_K) \in \mathbb{R}_+^{K-1}$ such that $U_k(a_k) = U_1(\overline{a})$ and $U_k^{'}(a_k) = U_1^{'}(\overline{a})$ for all $k = 2, \ldots, K$.

Before presenting our main propositions, we summarize Fleurbaey et al.' s theorem. Let $\succsim_{FHT}$ be the quasi-ordering which corresponds with welfare dominance according to all profiles in the family of utility profiles $(U_1, ..., U_K)$ satisfying assumptions A1, A2$_{\boldsymbol{\alpha}}$, A2$^{\boldsymbol{\beta}}$ and A5. This is a superset of $\mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ and therefore $\succsim_{FHT}$-dominance must imply $\succsim_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$-dominance.

Fleurbaey, Hagneré and Trannoy (2003). *Consider two heterogeneous distributions F and G as well as lower and upper bound vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^K$ with $\beta_k \geq \alpha_k \geq 1 = \beta_1 = \alpha_1$, for all $k \in \mathbb{K}$, and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. Define (for each $k \in \mathbb{K}$) a function H_k as in proposition 1. Setting $Z_K = H_K$, define functions Z_k recursively (starting from $k = K$ downwards to $k = 2$) as*

$$Z_{k-1} : y \mapsto H_{k-1}(y) + \max_{\alpha_k y \leq x \leq \beta_k y} \{Z_k(x)\}.$$

Now, $F \succsim_{FHT} G$ holds if and only if $Z_1(y) \leq 0$ holds for all $y \in [0, \overline{a}]$.

Our next proposition shows how welfare dominance for $\succsim_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ can be implemented; a discussion follows:

Proposition 3. *Consider two heterogeneous distributions F and G as well as lower and upper bound vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^K$ with $\beta_k \geq \alpha_k \geq 1 = \beta_1 = \alpha_1$, for all $k \in \mathbb{K}$, and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. Define an indicator function I which equals one if its argument is true, and zero otherwise.*
Sufficient conditions. Let $\underline{Z}_K = H_K$. Recursively define functions (starting from $k = K$ downwards to $k = 2$)

$$\underline{Z}_{k-1}(y) = H_{k-1}(y) + I\left(\underline{Z}_k(\alpha_k y) \leq 0\right) \underline{Z}_k(\alpha_k y) + I\left(y \leq \frac{\alpha_k^* \overline{a}}{\beta_k}\right) I\left(\underline{Z}_k(\beta_k y) \geq 0\right) \underline{Z}_k(\beta_k y).$$

We get

$$\text{if } \underline{Z}_1(y) \leq 0 \text{ for all } y \in [0, \overline{a}], \text{ then } F \succsim_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} G. \tag{4}$$

Necessary conditions. Let $\overline{Z}_K = H_K$. Recursively define functions (starting from $k = K$ downwards to $k = 2$)

$$\overline{Z}_{k-1}(y) = H_{k-1}(y) + I\left(\overline{Z}_k(\alpha_k y) \geq 0\right)\overline{Z}_k(\alpha_k y) + I\left(y \leq \frac{\alpha_k^* \overline{a}}{\beta_k}\right) I\left(\overline{Z}_k(\beta_k y) \leq 0\right)\overline{Z}_k(\beta_k y).$$

We get

$$\text{if } F \succ_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} G, \text{ then } \overline{Z}_1(y) < 0 \text{ for some } y \in [0, \overline{a}]. \tag{5}$$

First, the sufficient conditions $\underline{Z}_1(y) \leq 0$ *for all* $y \in [0, \overline{a}]$ can be used to check whether $F \succsim_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} G$ holds, while we can exclude the strict dominance case $F \succ_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} G$ whenever the necessary conditions are not satisfied, i.e., whenever $\overline{Z}_1(y) \geq 0$ holds for all $y \in [0, \overline{a}]$.

Second, in case the lower and upper bound vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ coincide, so that an equivalence scale $m_k = \alpha_k^* = \beta_k^*$ is in fact implied for the types k, the function values $\underline{Z}_1(y)$ and $\overline{Z}_1(y)$ are equal to $\sum_{k \in \mathbb{K}} H_k(m_k y)$. The sufficient conditions in this case yield

$$\sum_{k \in \mathbb{K}} H_k(m_k y) \leq 0 \text{ for all } y \in [0, \overline{a}]. \tag{6}$$

This corresponds with Ebert's (1999) proposal: checking whether F generalized Lorenz dominates G on the basis of household incomes, divided and weighted by equivalence scales.

Third, we look at the other extreme, i.e., when all β_k's (for $k = 2, \ldots, K$) approach infinity. The following corollary tells us that the conditions (4) yield the sequential conditions (2) of proposition 1 when the β_k's (for $k = 2, \ldots, K$) approach infinity. Additionally choosing $\boldsymbol{\alpha} = \mathbf{1}$ would lead to Jenkins and Lambert's (1993) extension of the sequential generalized Lorenz dominance conditions.[5]

Corollary 1. *Consider two heterogeneous distributions F and G as well as lower and upper bound vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^K$ with $\beta_k \geq \alpha_k \geq 1 = \alpha_1 = \beta_1$, for all $k \in \mathbb{K}$, and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. If $\boldsymbol{\beta} \to (1, \infty, \ldots, \infty)$, then the conditions in (4) become equivalent with $\sum_{i=k}^K H_i(\alpha_i^* y) \leq 0$, for all $y \in [0, \overline{a}]$ and for all $k \in \mathbb{K}$.*

Finally, we compare proposition 3 with Fleurbaey et al.'s (2003) criterion $\succsim_{FHT}$. Although Fleurbaey et al.'s (2003) criterion is implementable, the iterated maximum-procedure complicates things a lot in practice. In contrast, the functions $\underline{Z}_k$ and $\overline{Z}_k$ can

[5] In order to recover the conditions of proposition 2 from those of proposition 3, we would need to relax the restrictions imposed on α at the start of the paper to allow each α_k go to zero.

be easily calculated, and thus the conditions in proposition 3 are implementable in a straightforward way. Of course, this practical point would be worthless if our sufficient conditions in (4) would have less ranking power compared to Fleurbaey et al.'s criterion. Our final corollary 2 tells us that this is not the case: whenever F dominates G according to $\succsim_{FHT}$ the sufficient conditions (4) provided by proposition 3 are satisfied. Furthermore, whenever G dominates F according to $\succsim_{FHT}$, the necessary conditions (5) provided by proposition 3 cannot be satisfied.

Corollary 2. *Consider two heterogeneous distributions F and G as well as lower and upper bound vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^K$ with $\beta_k \geq \alpha_k \geq 1 = \beta_1 = \alpha_1$, for all $k \in \mathbb{K}$, and an exogeneous income level $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. If $F \succsim_{FHT} G$ holds, then also the sufficient conditions in (4) are satisfied. If $G \succsim_{FHT} F$ holds, then the necessary conditions in (5) cannot be satisfied.*

6 Conclusion

Atkinson and Bourguignon's (1987) welfare ordering in the case where the population is partitioned into subgroups on the basis of needs is a utilitarian criterion based on social utility functions which satisfy reasonable conditions, and it can be implemented by applying the SGLD criterion. This approach was devised as an alternative to invoking a specific equivalence scale to make a heterogeneous welfare comparison.

In Fleurbaey et al. (2003), it is argued inter alia that the SGLD criterion admits social utility profiles "considered unreasonable by all practitioners" (ibid., p. 311). An alternative equivalence-scale-based framework of analysis is advocated, in which (lower and upper) limits are placed on the relative needs of the different household types, by positing flexible equivalence scales bounded to lie within certain ranges. This leads to a dominance criterion and an algorithm for implementing it - thereby, Fleurbaey et al. argue, providing "a middleway criterion" between Ebert's (1997,1999) fixed equivalence scale approach and that of Bourguignon (1989).

In this paper we have extended the SGLD criterion, which is not predicated upon equivalence scales, by introducing lower and upper bounds directly into the household utility profiles, first separately, and then together. We have obtained new dominance criteria as the result of this refinement, which retain the character of SGLD and also relate well to the fixed equivalence scale approach of Ebert (1997,1999), though not being in any way dependent on the contentious equivalence scale methodology.

When lower bounds are introduced, necessary and sufficient sequential conditions arise, in terms of divided and weighted household incomes, starting from the most needy type, followed by the most and second most needy types, and so on. When upper bounds (only) are introduced, a reversed sequential procedure proves to be necessary and sufficient, starting with the least needy group (singles), followed by the two least needy groups taken together (singles and couples) and so on.

When lower and upper bounds are introduced simultaneously, separate necessary and sufficient criteria are determined, which are intermediate between Ebert's (1997,1999) equivalence scale weighted approach and Atkinson and Bourguignon's (1987) sequential GLD approach. Finally, compared with Fleurbaey et al.'s (2003) criterion, our sufficient conditions can be easily implemented and allow for a more complete ranking, which should be of interest to practitioners.

Appendix

Proof of proposition 1

Sufficiency: Given two distributions F and G, define functions $\widetilde{H}_k = p_k F_k - q_k G_k$ for all $k \in \mathbb{K}$, so that $H_k(y) = \int_0^y \widetilde{H}_k(x)\,dx$. The difference in welfare for a profile $U = (U_1, \ldots, U_K) \in \mathcal{U}_{\boldsymbol{\alpha}}$ equals:

$$\Delta W_U = \sum_{k\in\mathbb{K}} \int_{\underline{s}_k}^{\overline{s}_k} U_k(y)\,d\widetilde{H}_k(y) = \sum_{k\in\mathbb{K}} \int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k(y)\,d\widetilde{H}_k(y),$$

with $0 < \underline{a} < \min\left(\frac{\underline{s}_1}{\alpha_1^*}, \ldots, \frac{\underline{s}_K}{\alpha_K^*}\right)$ and $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. Using partial integration twice, together with the definition of H_k, a change of variable and assumption A4$_{\boldsymbol{\alpha}}$, we get

$$\begin{aligned}
\Delta W_U &= -\sum_{k\in\mathbb{K}} U_k^{'}(\alpha_k^*\overline{a})\,H_k(\alpha_k^*\overline{a}) + \sum_{k\in\mathbb{K}} \int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k^{''}(y)\,H_k(y)\,dy \\
&= \underbrace{-\sum_{k\in\mathbb{K}} U_k^{'}(\alpha_k^*\overline{a})\,H_k(\alpha_k^*\overline{a})}_{A} + \int_{\underline{a}}^{\overline{a}} \underbrace{\sum_{k\in\mathbb{K}} \alpha_k^* U_k^{''}(\alpha_k^* y)\,H_k(\alpha_k^* y)}_{B}\,dy
\end{aligned}$$

We can rewrite A and B as

$$\begin{aligned}
A &= -\underbrace{U_1^{'}(\overline{a})}_{\geq 0 \text{ via A1}} \sum_{i\in\mathbb{K}} H_i(\alpha_i^*\overline{a}) - \sum_{k=2}^{K} \underbrace{\left(U_k^{'}(\alpha_k^*\overline{a}) - U_{k-1}^{'}\left(\alpha_{k-1}^*\overline{a}\right)\right)}_{\geq 0 \text{ via A2}_{\boldsymbol{\alpha}}} \left(\sum_{i=k}^{K} H_i(\alpha_i^*\overline{a})\right) \\
B &= \underbrace{\alpha_1^* U_1^{''}(\alpha_1^* y)}_{\leq 0 \text{ via A1}} \sum_{i\in\mathbb{K}} H_i(\alpha_i^* y) + \sum_{k=2}^{K} \underbrace{\left(\alpha_k^* U_k^{''}(\alpha_k^* y) - \alpha_{k-1}^* U_{k-1}^{''}\left(\alpha_{k-1}^* y\right)\right)}_{\leq 0 \text{ via A3}_{\boldsymbol{\alpha}}} \left(\sum_{i=k}^{K} H_i(\alpha_i^* y)\right).
\end{aligned}$$

Therefore, sufficient conditions for welfare dominance are

$$\sum_{i=k}^{K} H_i(\alpha_i^* y) \leq 0, \text{ for all } y \in [0, \overline{a}] \text{ for all } k \in \mathbb{K}.$$

Necessity:

2. Suppose $\Delta W \geq 0$ for all utility profiles in $\mathcal{U}_{\boldsymbol{\alpha}}$, but there exists a k such that $\sum_{i=k}^{K} H_i(\alpha_i^* y) > 0$ on a certain non-degenerate income interval $[a, b]$, which belongs to $[0, \overline{a}]$; this is only possible if $\min\left(\frac{\underline{s}_k}{\alpha_k^*}, \ldots, \frac{\underline{s}_K}{\alpha_K^*}\right) \leq a$, otherwise $\sum_{i=k}^{K} H_i(\alpha_i^* a) = 0$. Choose a utility profile $(U_1, \ldots, U_K)$ consisting of twice continuously differentiable utility functions such that

$$\begin{cases} U_1 = \ldots = U_{k-1} : y \mapsto 0. \\ U_k : \begin{cases} U_k^{'} = C > 0 \text{ (and thus } U_k^{''} = 0\text{), for } y \leq \alpha_k^* a \\ U_k^{''} < 0, \text{ for } \alpha_k^* a < y < \alpha_k^* b \\ U_k = 0 \text{ (and thus } U_k^{'} = U_k^{''} = 0\text{), for } y \geq \alpha_k^* b \end{cases}, \\ U_{k+1} : y \mapsto \alpha_{k+1} U_k\left(\frac{y}{\alpha_{k+1}}\right) = \frac{\alpha_{k+1}^*}{\alpha_k^*} U_k\left(\frac{\alpha_k^*}{\alpha_{k+1}^*} y\right), \\ U_{k+2} : y \mapsto \alpha_{k+2} U_{k+1}\left(\frac{y}{\alpha_{k+2}}\right) = \alpha_{k+2}\alpha_{k+1} U_k\left(\frac{y}{\alpha_{k+2}\alpha_{k+1}}\right) = \frac{\alpha_{k+2}^*}{\alpha_k^*} U_k\left(\frac{\alpha_k^*}{\alpha_{k+2}^*} y\right), \\ \ldots \\ U_K : y \mapsto \frac{\alpha_K^*}{\alpha_k^*} U_k\left(\frac{\alpha_k^*}{\alpha_K^*} y\right), \end{cases}$$

This profile belongs to $\mathcal{U}_{\boldsymbol{\alpha}}$. Choosing $\underline{a} \leq \min\left(\frac{\underline{s}_k}{\alpha_k^*}, \ldots, \frac{\underline{s}_K}{\alpha_K^*}\right) \leq a$, the welfare difference for this profile equals

$$\Delta W_U = \sum_{k \in \mathbb{K}} \int_{\underline{s}_k}^{\overline{s}_k} U_k(y)\, d\widetilde{H}_k(y) = \sum_{i=k}^{K} \int_{\alpha_i^* \underline{a}}^{\alpha_i^* b} U_i(y)\, d\widetilde{H}_i(y)$$

and using partial integration twice and replacing $U_i^{''}$ (first step) and a change of variable (second step), we can rewrite ΔW_U as

$$\begin{aligned} \Delta W_U &= \sum_{i=k}^{K} \int_{\alpha_i^* \underline{a}}^{\alpha_i^* b} U_i^{''}(y)\, H_i(y)\, dy = \sum_{i=k}^{K} \int_{\alpha_i^* a}^{\alpha_i^* b} U_i^{''}(y)\, H_i(y)\, dy \\ &= \sum_{i=k}^{K} \int_{\alpha_i^* a}^{\alpha_i^* b} \frac{\alpha_k^*}{\alpha_i^*} U_k^{''}\left(\frac{\alpha_k^*}{\alpha_i^*} y\right) H_i(y)\, dy = \sum_{i=k}^{K} \int_a^b \alpha_k^* U_k^{''}(\alpha_k^* y)\, H_i(\alpha_i^* y)\, dy \\ &= \int_a^b \underbrace{\alpha_k^* U_k^{''}(\alpha_k^* y)}_{<0 \text{ on } [a,b]} \underbrace{\sum_{i=k}^{K} H_i(\alpha_i^* y)\, dy}_{>0 \text{ on } [a,b]} \end{aligned}$$

which is strictly negative and thus contradicts $\Delta W_U \geq 0$.

Proof of proposition 2

Sufficiency: With F, G and $\widetilde{H}_k$ as defined before, the difference in welfare for a profile $U = (U_1, \ldots, U_K) \in \mathcal{U}^{\boldsymbol{\beta}}$ equals:

$$\Delta W_U = \sum_{k \in \mathbb{K}} \int_{\underline{s}_k}^{\overline{s}_k} U_k(y)\, d\widetilde{H}_k(y) = \sum_{k \in \mathbb{K}} \int_{\beta_k^* \underline{a}}^{\beta_k^* \overline{a}} U_k(y)\, d\widetilde{H}_k(y),$$

with $0 < \underline{a} < \min\left(\frac{\underline{s}_1}{\beta_1^*}, \ldots, \frac{\underline{s}_K}{\beta_K^*}\right)$ and $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\beta_1^*}, \ldots, \frac{\overline{s}_K}{\beta_K^*}\right)$. Using partial integration twice, together with the definition of H_k, a change of variable and assumption A4$^{\boldsymbol{\beta}}$, we get

$$\begin{aligned}
\Delta W_U &= -\sum_{k\in\mathbb{K}} U_k^{'}\left(\beta_k^*\overline{a}\right) H_k\left(\beta_k^*\overline{a}\right) + \sum_{k\in\mathbb{K}} \int_{\beta_k^*\underline{a}}^{\beta_k^*\overline{a}} U_k^{''}\left(y\right) H_k\left(y\right) dy \\
&= \underbrace{-\sum_{k\in\mathbb{K}} U_k^{'}\left(\beta_k^*\overline{a}\right) H_k\left(\beta_k^*\overline{a}\right)}_{A} + \int_{\underline{a}}^{\overline{a}} \underbrace{\sum_{k\in\mathbb{K}} \beta_k^* U_k^{''}\left(\beta_k^* y\right) H_k\left(\beta_k^* y\right) dy}_{B}
\end{aligned}$$

We can rewrite A and B as

$$\begin{aligned}
A &= -\underbrace{U_K^{'}\left(\overline{a}\right)}_{\geq 0 \text{ via A1}} \sum_{i\in\mathbb{K}} H_i\left(\beta_i^*\overline{a}\right) - \sum_{k=2}^{K} \underbrace{\left(U_{k-1}^{'}\left(\beta_{k-1}^*\overline{a}\right) - U_k^{'}\left(\beta_k^*\overline{a}\right)\right)}_{\geq 0 \text{ via A2}^{\boldsymbol{\beta}}} \left(\sum_{i=1}^{k-1} H_i\left(\beta_i^*\overline{a}\right)\right) \\
B &= \underbrace{\beta_K^* U_K^{''}\left(\beta_K^* y\right)}_{\leq 0 \text{ via A1}} \sum_{i\in\mathbb{K}} H_i\left(\beta_i^* y\right) + \sum_{k=2}^{K} \underbrace{\left(\beta_{k-1}^* U_{k-1}^{''}\left(\beta_{k-1}^* y\right) - \beta_k^* U_k^{''}\left(\beta_k^* y\right)\right)}_{\leq 0 \text{ via A3}^{\boldsymbol{\beta}}} \left(\sum_{i=1}^{k-1} H_i\left(\beta_i^* y\right)\right).
\end{aligned}$$

Therefore, sufficient conditions for welfare dominance are

$$\sum_{i=1}^{k} H_i\left(\beta_i^* y\right) \leq 0, \text{ for all } y \in [0, \overline{a}] \text{ for all } k \in \mathbb{K}.$$

Necessity:
Suppose $\Delta W \geq 0$ for all utility profiles in $\mathcal{U}^{\boldsymbol{\beta}}$, but there exists a k such that $\sum_{i=1}^{k} H_i\left(\beta_i^* a\right) > 0$ on a certain non-degenerate income interval $[a, b]$, which belongs to $[0, \overline{a}]$; this is only possible if $\min\left(\frac{\underline{s}_1}{\beta_1^*}, \ldots, \frac{\underline{s}_k}{\beta_k^*}\right) \leq a$, otherwise $\sum_{i=1}^{k} H_i\left(\beta_i^* a\right) = 0$. Choose a utility profile $(U_1, \ldots, U_K)$ which is twice continuously differentiable and satisfies:

$$\begin{cases}
U_1 : \begin{cases} U_1^{'} = C > 0 \text{ (thus } U_1^{''} = 0\text{), for all } y \leq a \\ U_1^{''} < 0, \text{ for } a < y < b \\ U_1 = 0 \text{ (and thus } U_1^{'} = U_1^{''} = 0\text{) for } y \geq b \end{cases} , \\
U_2 : y \mapsto \beta_2 U_1\left(\frac{y}{\beta_2}\right) = \beta_2^* U_1\left(\frac{y}{\beta_2^*}\right), \\
U_3 : y \mapsto \beta_3 U_2\left(\frac{y}{\beta_3}\right) = \beta_3^* U_1\left(\frac{y}{\beta_3^*}\right), \\
\ldots \\
U_k : y \mapsto \beta_k U_{k-1}\left(\frac{y}{\beta_k}\right) = \ldots = \beta_k^* U_1\left(\frac{y}{\beta_k^*}\right), \\
U_{k+1} = \ldots = U_K : y \mapsto 0.
\end{cases}$$

This profile belongs to $\mathcal{U}^{\boldsymbol{\beta}}$. Choosing $\underline{a} \leq \min\left(\frac{\underline{s}_1}{\beta_1^*}, \ldots, \frac{\underline{s}_k}{\alpha_k^*}\right) \leq a$, the welfare difference for this profile equals

$$\Delta W_U = \sum_{k\in\mathbb{K}} \int_{\underline{s}_k}^{\overline{s}_k} U_k\left(y\right) d\widetilde{H}_k\left(y\right) = \sum_{i=1}^{k} \int_{\beta_i^*\underline{a}}^{\beta_i^* b} U_i\left(y\right) d\widetilde{H}_i\left(y\right)$$

and using partial integration twice and replacing $U_i^{''}$ (first step) and a change of variable (second step), we can rewrite ΔW_U as

$$\begin{aligned}\Delta W_U &= \sum_{i=1}^{k}\int_{\beta_i^*\underline{a}}^{\beta_i^* b} U_i^{''}(y)\,H_i(y)\,dy = \sum_{i=1}^{k}\int_{\beta_i^* a}^{\beta_i^* b} U_i^{''}(y)\,H_i(y)\,dy \\ &= \sum_{i=1}^{k}\int_{\beta_i^* a}^{\beta_i^* b} \frac{1}{\beta_i^*}U_1^{''}\left(\frac{y}{\beta_i^*}\right)H_i(y)\,dy = \sum_{i=1}^{k}\int_{a}^{b} U_1^{''}(y)\,H_i(\beta_i^* y)\,dy \\ &= \int_a^b \underbrace{U_1^{''}(y)}_{<0 \text{ on } [a,b]} \underbrace{\sum_{i=1}^{k} H_i(\beta_i^* y)}_{>0 \text{ on } [a,b]}\,dy,\end{aligned}$$

which is strictly negative and thus contradicts $\Delta W_U \geq 0$.

Proof of proposition 3

We focus on the case $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$; in case $\boldsymbol{\alpha} = \boldsymbol{\beta}$, the proof is the same as for the generalized Lorenz dominance criterion. As in the proof of Proposition 2, the difference in welfare for a profile $U = (U_1, \ldots, U_K) \in \mathcal{U}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ equals

$$\Delta W_U = \sum_{k\in\mathbb{K}}\int_{\underline{s}_k}^{\overline{s}_k} U_k(y)\,d\widetilde{H}_k(y) = \sum_{k\in\mathbb{K}}\int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k(y)\,d\widetilde{H}_k(y),$$

with $0 < \underline{a} < \min\left(\frac{\underline{s}_1}{\alpha_1^*}, \ldots, \frac{\underline{s}_K}{\alpha_K^*}\right)$ and $\overline{a} \geq \max\left(\frac{\overline{s}_1}{\alpha_1^*}, \ldots, \frac{\overline{s}_K}{\alpha_K^*}\right)$. Notice that, if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, A4$_{\boldsymbol{\alpha}}$ and A4$^{\boldsymbol{\beta}}$ together with A2$_{\boldsymbol{\alpha}}$ imply $U_k^{'}(\alpha_k^*\overline{a}) = 0$, for all $k \in \mathbb{K}$. Using partial integration twice, together with the definition of H_k, we get

$$\Delta W_U = \sum_{k\in\mathbb{K}}\int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k^{''}(y)\,H_k(y)\,dy.$$

3A. We start with the sufficient conditions. Rewrite A3$_{\boldsymbol{\alpha}}$ and A3$^{\boldsymbol{\beta}}$ for $k = K$ as

$$\frac{1}{\beta_K}U_{K-1}^{''}\left(\frac{y}{\beta_K}\right) \leq U_K^{''}(y) \leq \frac{1}{\alpha_K}U_{K-1}^{''}\left(\frac{y}{\alpha_K}\right).$$

Let $\underline{Z}_K = H_K$ and define an indicator function I which equals 1 if its argument is true and zero otherwise. We have

$$\int_{\alpha_K^*\underline{a}}^{\alpha_K^*\overline{a}} U_K^{''}(y)\,H_K(y)\,dy$$

$$\begin{aligned}&\geq \int_{\alpha_K^*\underline{a}}^{\alpha_K^*\overline{a}} \tfrac{1}{\alpha_K}U_{K-1}^{''}\left(\tfrac{y}{\alpha_K}\right) I\left(\underline{Z}_K(y) \leq 0\right)\underline{Z}_K(y)\,dy + \int_{\alpha_K^*\underline{a}}^{\alpha_K^*\overline{a}} \tfrac{1}{\beta_K}U_{K-1}^{''}\left(\tfrac{y}{\beta_K}\right) I\left(\underline{Z}_K(y) \geq 0\right)\underline{Z}_K(y)\,dy \\ &= \int_{\alpha_{K-1}^*\underline{a}}^{\alpha_{K-1}^*\overline{a}} U_{K-1}^{''}(y)\,I\left(\underline{Z}_K(\alpha_K y) \leq 0\right)\underline{Z}_K(\alpha_K y)\,dy + \int_{\frac{\alpha_K^*\underline{a}}{\beta_K}}^{\frac{\alpha_K^*\overline{a}}{\beta_K}} U_{K-1}^{''}(y)\,I\left(\underline{Z}_K(\beta_K y) \geq 0\right)\underline{Z}_K(\beta_K y)\,dy \\ &= \int_{\frac{\alpha_K^*\underline{a}}{\beta_K}}^{\alpha_{K-1}^*\overline{a}} U_{K-1}^{''}(y)\left[I\left(\underline{Z}_K(\alpha_K y) \leq 0\right)\underline{Z}_K(\alpha_K y) + I\left(y \leq \frac{\alpha_K^*\overline{a}}{\beta_K}\right) I\left(\underline{Z}_K(\beta_K y) \geq 0\right)\underline{Z}_K(\beta_K y)\right] dy,\end{aligned}$$

where the first equality follows by a change of variable, while the second equality follows because $\underline{Z}_K(\alpha_K y) = 0$ for all incomes $\frac{\alpha_K^*}{\beta_K}\underline{a} \leq y \leq \alpha_{K-1}^*\underline{a}$. Defining

$$\underline{Z}_{K-1}(y) = H_{K-1}(y) + I\left(\underline{Z}_K(\alpha_K y) \leq 0\right)\underline{Z}_K(\alpha_K y) + I\left(y \leq \frac{\alpha_K^*\overline{a}}{\beta_K}\right) I\left(\underline{Z}_K(\beta_K y) \geq 0\right)\underline{Z}_K(\beta_K y),$$

we get (given that $H_{K-1}(y) = 0$ for all incomes $\frac{\alpha_K^*}{\beta_K}\underline{a} \leq y \leq \alpha_{K-1}^*\underline{a}$)

$$\sum_{k=K-1}^{K}\int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k''(y)\, H_k(y)\, dy \geq \int_{\frac{\alpha_K^*}{\beta_K}\underline{a}}^{\alpha_{K-1}^*\overline{a}} U_{K-1}''(y)\, \underline{Z}_{K-1}(y)\, dy.$$

In the same way as before, a lower bound for the right-hand side can be obtained as

$$\int_{\frac{\alpha_K^*}{\beta_K}\underline{a}}^{\alpha_{K-1}^*\overline{a}} U_{K-1}''(y)\, \underline{Z}_{K-1}(y)\, dy$$

$$\geq \int_{\frac{\alpha_K^*\underline{a}}{\beta_K\beta_{K-1}}}^{\alpha_{K-2}^*\overline{a}} U_{K-2}''(y)\left(\underline{Z}_{K-2}(y) - H_{K-2}(y)\right) dy$$

with

$$\begin{aligned}\underline{Z}_{K-2}(y) &= H_{K-2}(y) + I\left(\underline{Z}_{K-1}(\alpha_{K-1}y) \leq 0\right)\underline{Z}_{K-1}(\alpha_{K-1}y) + \\ &\quad I\left(y \leq \frac{\alpha_{K-1}^*\overline{a}}{\beta_{K-1}}\right) I\left(\underline{Z}_{K-1}(\beta_{K-1}y) \geq 0\right)\underline{Z}_{K-1}(\beta_{K-1}y).\end{aligned}$$

Given that $H_{K-2}(y) = 0$ for incomes $y \leq \frac{\alpha_K^*\underline{a}}{\beta_K\beta_{K-1}} \leq \alpha_{K-2}^*\underline{a}$, we get

$$\sum_{k=K-2}^{K}\int_{\alpha_k^*\underline{a}}^{\alpha_k^*\overline{a}} U_k''(y)\, H_k(y)\, dy \geq \int_{\frac{\alpha_K^*\underline{a}}{\beta_K\beta_{K-1}}}^{\alpha_{K-2}^*\overline{a}} U_{K-2}''(y)\, \underline{Z}_{K-2}(y)\, dy.$$

Proceeding in this way, we end up with a lower bound for ΔW_U, i.e.,

$$\Delta W_U \geq \int_{\frac{\alpha_K^*}{\beta_K^*}\underline{a}}^{\overline{a}} U_1''(y)\, \underline{Z}_1(y)\, dy,$$

with

$$\underline{Z}_1(y) = H_1(y) + I\left(\underline{Z}_2(\alpha_2 y) \leq 0\right)\underline{Z}_2(\alpha_2 y) + I\left(y \leq \frac{\alpha_2^*\overline{a}}{\beta_2}\right) I\left(\underline{Z}_2(\beta_2 y) \geq 0\right)\underline{Z}_2(\beta_2 y).$$

Because $\underline{Z}_1(y) = 0$ on $\left[0, \frac{\alpha_K^*}{\beta_K^*}\underline{a}\right]$, sufficiency of $\underline{Z}_1(y) \leq 0$ for all $y \in [0, \overline{a}]$ is straightforward.

3B. Let us now focus on necessary conditions. Choose $\overline{Z}_K = H_K$. In the same way as in (3A) we get

$$\int_{\alpha_K^*\underline{a}}^{\alpha_K^*\overline{a}} U_K''(y)\, H_K(y)\, dy$$

$$
\begin{aligned}
&\leq \int_{\alpha_K^* \underline{a}}^{\alpha_K^* \overline{a}} \tfrac{1}{\beta_K} U_{K-1}'' \left(\tfrac{y}{\beta_K}\right) I\left(\overline{Z}_K(y) \leq 0\right) \overline{Z}_K(y)\, dy + \int_{\alpha_K^* \underline{a}}^{\alpha_K^* \overline{a}} \tfrac{1}{\alpha_K} U_{K-1}'' \left(\tfrac{y}{\alpha_K}\right) I\left(\overline{Z}_K(y) \geq 0\right) \overline{Z}_K(y)\, dy \\
&= \int_{\frac{\alpha_K^* \underline{a}}{\beta_K}}^{\frac{\alpha_K^* \overline{a}}{\beta_K}} U_{K-1}''(y)\, I\left(\overline{Z}_K(\beta_K y) \leq 0\right) \overline{Z}_K(\beta_K y)\, dy + \int_{\alpha_{K-1}^* \underline{a}}^{\alpha_{K-1}^* \overline{a}} U_{K-1}''(y)\, I\left(\overline{Z}_K(\alpha_K y) \geq 0\right) \overline{Z}_K(\alpha_K y)\, dy \\
&= \int_{\frac{\alpha_K^* \underline{a}}{\beta_K}}^{\alpha_{K-1}^* \overline{a}} U_{K-1}''(y) \left[I\left(y \leq \tfrac{\alpha_K^* \overline{a}}{\beta_K}\right) I\left(\overline{Z}_K(\beta_K y) \leq 0\right) \overline{Z}_K(\beta_K y) + I\left(\overline{Z}_K(\alpha_K y) \geq 0\right) \overline{Z}_K(\alpha_K y) \right] dy,
\end{aligned}
$$

where the first equality follows by a change of variable, while the second equality follows because $\overline{Z}_K(\alpha_K y) = 0$ for all incomes $\frac{\alpha_K^*}{\beta_K} \underline{a} \leq y \leq \alpha_{K-1}^* \underline{a}$. Defining

$$
\overline{Z}_{K-1}(y) = H_{K-1}(y) + I\left(y \leq \frac{\alpha_K^* \overline{a}}{\beta_K}\right) I\left(\overline{Z}_K(\beta_K y) \leq 0\right) \overline{Z}_K(\beta_K y) + I\left(\overline{Z}_K(\alpha_K y) \geq 0\right) \overline{Z}_K(\alpha_K y),
$$

we get (given that $H_{K-1}(y) = 0$ for all incomes $\frac{\alpha_K^*}{\beta_K} \underline{a} \leq y \leq \alpha_{K-1}^* \underline{a}$)

$$
\sum_{k=K-1}^{K} \int_{\alpha_k^* \underline{a}}^{\alpha_k^* \overline{a}} U_k''(y)\, H_k(y)\, dy \leq \int_{\frac{\alpha_K^*}{\beta_K} \underline{a}}^{\alpha_{K-1}^* \overline{a}} U_{K-1}''(y)\, \overline{Z}_{K-1}(y)\, dy.
$$

In the same way as before, an upper bound for the right-hand side can be obtained as

$$
\begin{aligned}
&\int_{\frac{\alpha_K^*}{\beta_K} \underline{a}}^{\alpha_{K-1}^* \overline{a}} U_{K-1}''(y)\, \overline{Z}_{K-1}(y)\, dy \\
&\leq \int_{\frac{\alpha_K^* \underline{a}}{\beta_K \beta_{K-1}}}^{\alpha_{K-2}^* \overline{a}} U_{K-2}''(y) \left(\overline{Z}_{K-2}(y) - H_{K-2}(y)\right) dy
\end{aligned}
$$

with

$$
\begin{aligned}
\overline{Z}_{K-2}(y) \quad = \quad & H_{K-2}(y) + I\left(y \leq \frac{\alpha_{K-1}^* \overline{a}}{\beta_{K-1}}\right) \left(\overline{Z}_{K-1}\left(\beta_{K-1} y\right) \leq 0\right) \overline{Z}_K\left(\beta_{K-1} y\right) + \\
& I\left(\overline{Z}_{K-1}(\alpha_{K-1} y) \geq 0\right) \overline{Z}_{K-1}(\alpha_{K-1} y),
\end{aligned}
$$

Given that $H_{K-2}(y) = 0$ for incomes $y \leq \frac{\alpha_K^* \underline{a}}{\beta_K \beta_{K-1}} \leq \alpha_{K-2}^* \underline{a}$, we get

$$
\sum_{k=K-2}^{K} \int_{\alpha_k^* \underline{a}}^{\alpha_k^* \overline{a}} U_k''(y)\, H_k(y)\, dy \leq \int_{\frac{\alpha_K^* \underline{a}}{\beta_K \beta_{K-1}}}^{\alpha_{K-2}^* \overline{a}} U_{K-2}''(y)\, \overline{Z}_{K-2}(y)\, dy.
$$

Proceeding in this way, we end up with an upper bound for ΔW_U, more precisely

$$
\Delta W_U \leq \int_{\frac{\alpha_K^*}{\beta_K^*} \underline{a}}^{\overline{a}} U_1''(y)\, \overline{Z}_1(y)\, dy,
$$

with

$$
\overline{Z}_1(y) = H_1(y) + I\left(\overline{Z}_2(\alpha_2 y) \geq 0\right) \overline{Z}_2(\alpha_2 y) + I\left(y \leq \frac{\alpha_2^* \overline{a}}{\beta_2}\right) I\left(\overline{Z}_2(\beta_2 y) \leq 0\right) \overline{Z}_2(\beta_2 y).
$$

Because $\overline{Z}_1$ equals zero on $\left[0, \frac{\alpha_K^*}{\beta_K^*} \underline{a}\right]$, necessity of $\overline{Z}_1(y) < 0$ for some $y \in [0, \overline{a}]$ is obvious.

Proof of corollary 1

Let $\boldsymbol{\beta} \rightarrow (1, \infty, \ldots, \infty)$. The sufficient condition in (4) can be rewritten (by changing variable) as

$$H_1\left(\frac{y}{\beta_K^*}\right)+I\left(\underline{Z}_2\left(\frac{\alpha_2 y}{\beta_K^*}\right)\leq 0\right)\underline{Z}_2\left(\frac{\alpha_2 y}{\beta_K^*}\right)+I\left(\frac{y}{\beta_K^*}\leq\frac{\alpha_2^*\overline{a}}{\beta_2}\right)I\left(\underline{Z}_2\left(\frac{\beta_2 y}{\beta_K^*}\right)\geq 0\right)\underline{Z}_2\left(\frac{\beta_2 y}{\beta_K^*}\right)\leq 0$$

for all $y \in [0, \beta_K^* \overline{a}]$. In the limit (i) the first two terms at the left-hand side equal zero and (ii) $I\left(\frac{y}{\beta_K^*} \leq \frac{\alpha_2^*\overline{a}}{\beta_2}\right) = 1 = I\left(\underline{Z}_2\left(\frac{\beta_2 y}{\beta_K^*}\right) \geq 0\right)$; thus, we proceed with the third term only and get as the sole sufficient condition when $\boldsymbol{\beta} \rightarrow (1, \infty, \ldots, \infty)$

$$\underline{Z}_2\left(\frac{\beta_2 y}{\beta_K^*}\right) \leq 0, \text{ for all } y \in [0, \beta_K^* \overline{a}],$$

or equivalently

$$H_2\left(\frac{\beta_2^* y}{\beta_K^*}\right)+I\left(\underline{Z}_3\left(\frac{\beta_2\alpha_3 y}{\beta_K^*}\right)\leq 0\right)\underline{Z}_3\left(\frac{\beta_2\alpha_3 y}{\beta_K^*}\right)+I\left(\frac{\beta_2 y}{\beta_K^*}\leq\frac{\alpha_3^*\overline{a}}{\beta_3}\right)I\left(\underline{Z}_3\left(\frac{\beta_3^* y}{\beta_K^*}\right)\geq 0\right)\underline{Z}_3\left(\frac{\beta_3^* y}{\beta_K^*}\right)\leq 0,$$

for all $y \in [0, \beta_K^* \overline{a}]$. Simplifying this term as before (at the limit, some terms equal zero, while others equal 1), we get

$$\underline{Z}_3\left(\frac{\beta_3^* y}{\beta_K^*}\right) \leq 0, \text{ for all } y \in [0, \beta_K^* \overline{a}] .$$

Proceeding in this way, and using $\underline{Z}_K = H_K$ and $\alpha_K^* \leq \beta_K^*$, we end up with

$$\begin{aligned} I\left(y \leq \alpha_K^* \overline{a}\right) I\left(\underline{Z}_K(y) \geq 0\right) \underline{Z}_K(y) &\leq 0, \text{ for all } y \in [0, \beta_K^* \overline{a}], \\ I\left(H_K(y) \geq 0\right) H_K(y) &\leq 0, \text{ for all } y \in [0, \alpha_K^* \overline{a}], \end{aligned}$$

which is possible if and only if $H_K(y) \leq 0$, for all $y \in [0, \alpha_K^* \overline{a}]$, or equivalently $H_K(\alpha_K^* y) \leq 0$, for all $y \in [0, \overline{a}]$; call this condition (*). It is possible to repeat this exercise, by a different change of variable, e.g.,

$$\begin{aligned} H_1\left(\frac{y}{\beta_{K-1}^*}\right)+I\left(\underline{Z}_2\left(\frac{\alpha_2 y}{\beta_{K-1}^*}\right)\leq 0\right)\underline{Z}_2\left(\frac{\alpha_2 y}{\beta_{K-1}^*}\right)+ \\ +I\left(\frac{y}{\beta_{K-1}^*}\leq\frac{\alpha_2^*\overline{a}}{\beta_2}\right)I\left(\underline{Z}_2\left(\frac{\beta_2 y}{\beta_{K-1}^*}\right)\geq 0\right)\underline{Z}_2\left(\frac{\beta_2 y}{\beta_{K-1}^*}\right)\leq 0, \end{aligned}$$

for all $y \in \left[0, \beta_{K-1}^* \overline{a}\right]$. This would lead us to

$$I\left(y \leq \alpha_{K-1}^* \overline{a}\right) I\left(\underline{Z}_{K-1}(y) \geq 0\right) \underline{Z}_{K-1}(y) \leq 0,$$

for all $y \in \left[0, \beta_{K-1}^* \overline{a}\right]$, or equivalently (using $\alpha_{K-1}^* \leq \beta_{K-1}^*$ and $\underline{Z}_K = H_K$)

$$I\left(H_{K-1}(y)+I\left(H_K(\alpha_K y)\leq 0\right)H_K(\alpha_K y)\geq 0\right)\left(H_{K-1}(y)+I\left(H_K(\alpha_K y)\leq 0\right)H_K(\alpha_K y)\right)\leq 0,$$

for all $y \in \left[0, \alpha^*_{K-1}\overline{a}\right]$. Changing variable in (*), we know that $H_K(\alpha_K y) \leq 0$, for all $y \in \left[0, \alpha^*_{K-1}\overline{a}\right]$, thus $I(H_K(\alpha_K y) \leq 0) = 1$. Thus, we get

$$I(H_{K-1}(y) + H_K(\alpha_K y) \geq 0)(H_{K-1}(y) + H_K(\alpha_K y)) \leq 0,$$

for all $y \in \left[0, \alpha^*_{K-1}\overline{a}\right]$, which is possible if and only if $H_{K-1}(y) + H_K(\alpha_K y) \leq 0$, for all $y \in \left[0, \alpha^*_{K-1}\overline{a}\right]$ or equivalently $H_{K-1}\left(\alpha^*_{K-1}y\right) + H_K(\alpha^*_K y) \leq 0$, for all $y \in [0, \overline{a}]$. Proceeding in this way, we can derive all sequential conditions defined in proposition 1.

Proof of corollary 2

2A. First, we prove that for an arbitrary (bounded) function f and scalars $1 \leq \alpha \leq \beta$, we have (for all $y \in \mathbb{R}_+$) $I(f(\alpha y) \leq 0) f(\alpha y) + I(f(\beta y) \geq 0) f(\beta y) \leq \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$; call it condition (*). In case $\alpha = \beta$, condition (*) is obvious. For $\alpha \neq \beta$, suppose $I(f(\alpha y) \leq 0) f(\alpha y) + I(f(\beta y) \geq 0) f(\beta y) > \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$, for some $y \in \mathbb{R}_+$. There are four cases to consider. The case $I(f(\alpha y) \leq 0) = 0$, $I(f(\beta y) \geq 0) = 0$ would lead to $f(\alpha y) > 0 > \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$, a contradiction. The cases $I(f(\alpha y) \leq 0) = 1$, $I(f(\beta y) \geq 0) = 0$ and $I(f(\alpha y) \leq 0) = 0$, $I(f(\beta y) \geq 0) = 1$ lead to (the contradictions) $f(\alpha y) > \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$ and $f(\beta y) > \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$, respectively. The final case $I(f(\alpha y) \leq 0) = 1$, $I(f(\beta y) \geq 0) = 1$ tells us that $f(\alpha y) \leq 0$, $f(\beta y) \geq 0$ and thus also $f(\beta y) \geq f(\alpha y) + f(\beta y) > \sup\limits_{\alpha y \leq x \leq \beta y} f(x)$, a contradiction again.

Second, we use condition (*) to prove that $Z_1 \geq \underline{Z}_1$. If so, we obtain that whenever $Z_1 \leq 0$ also $\underline{Z}_1 \leq 0$ holds, as required. We have (for all $y \in \mathbb{R}_+$) that

$$\begin{aligned}
Z_{K-1}(y) &= H_{K-1}(y) + \max_{\alpha_K y \leq x \leq \beta_K y} Z_K(x) \\
&= H_{K-1}(y) + \sup_{\alpha_K y \leq x \leq \beta_K y} H_K(x) \\
&\geq H_{K-1}(y) + I(\underline{Z}_K(\alpha_K y) \leq 0)\underline{Z}_K(\alpha_K y) + I(\underline{Z}_K(\beta_K y) \geq 0)\underline{Z}_K(\beta_K y) \\
&\geq \underbrace{H_{K-1}(y) + I(\underline{Z}_K(\alpha_K y) \leq 0)\underline{Z}_K(\alpha_K y) + I\left(y \leq \frac{\alpha^*_K \overline{a}}{\beta_K}\right) I(\underline{Z}_K(\beta_K y) \geq 0)\underline{Z}_K(\beta_K y)}_{\underline{Z}_{K-1}(y)}
\end{aligned}$$

where (i) the second equality follows from the definition $\underline{Z}_K = H_K$ and the fact that the maximum exists (and thus equals the supremum), (ii) the first inequality uses (*), and (iii) the second inequality comes from the fact that $I(\underline{Z}_K(\beta_K y) \geq 0)\underline{Z}_K(\beta_K y) \geq 0$ everywhere. Taking the supremum (which equals the maximum, if the latter exists) on both sides, we get

$$\max_{\alpha_{K-1} y \leq x \leq \beta_{K-1} y} Z_{K-1}(y) \geq \sup_{\alpha_{K-1} y \leq x \leq \beta_{K-1} y} \underline{Z}_{K-1}(y).$$

Using (*) again for the right-hand side, adding $H_{K-2}(y)$ to both sides and using the definitions of $Z_{K-2}(y)$ and $\underline{Z}_{K-2}(y)$, we obtain $Z_{K-2}(y) \geq \underline{Z}_{K-2}(y)$ (for all $y \in \mathbb{R}_+$). Proceeding this way, we end up with $Z_1(y) \geq \underline{Z}_1(y)$ (for all $y \in \mathbb{R}_+$), as required.

2B. Second, replacing f by $-f$ in (*) we have (for all $y \in \mathbb{R}_+$) $I(f(\alpha y) \geq 0) f(\alpha y) + I(f(\beta y) \leq 0) f(\beta y) \geq \inf_{\alpha y \leq x \leq \beta y} f(x)$; call it condition (**). In the same way as before, we deduce (for all $y \in \mathbb{R}_+$) that

$$\begin{aligned} & H_{K-1}(y) + \min_{\alpha_K y \leq x \leq \beta_K y} H_K(x) \\ \leq \; & H_{K-1}(y) + I\left(\overline{Z}_K(\alpha_K y) \geq 0\right) \overline{Z}_K(\alpha_K y) + I\left(\overline{Z}_K(\beta_K y) \leq 0\right) \overline{Z}_K(\beta_K y) \\ \leq \; & \underbrace{H_{K-1}(y) + I\left(\overline{Z}_K(\alpha_K y) \geq 0\right) \overline{Z}_K(\alpha_K y) + I\left(y \leq \frac{\alpha_K^* \overline{a}}{\beta_K}\right) I\left(\overline{Z}_K(\beta_K y) \leq 0\right) \overline{Z}_K(\beta_K y)}_{\overline{Z}_{K-1}(y)} \end{aligned}$$

where (i) the first inequality uses (**) and (ii) the second inequality follows from the fact that $I\left(\overline{Z}_K(\beta_K y) \geq 0\right) \overline{Z}_K(\beta_K y) \leq 0$ everywhere. Let $Z_{K-1}^{\circ}(y) = H_{K-1}(y) + \min_{\alpha_K y \leq x \leq \beta_K y} H_K(x)$. Taking the infimum (which equals the minimum, if the latter exists) on both sides, we get

$$\min_{\alpha_{K-1} y \leq x \leq \beta_{K-1} y} Z_{K-1}^{\circ}(y) \leq \inf_{\alpha_{K-1} y \leq x \leq \beta_{K-1} y} \overline{Z}_{K-1}(y).$$

Using (**) again for the right-hand side, adding $H_{K-2}(y)$ to both sides and using the definition of $\overline{Z}_{K-2}(y)$ and defining $Z_{K-2}^{\circ}(y) = H_{K-2}(y) + \min_{\alpha_{K-1} y \leq x \leq \beta_{K-1} y} Z_{K-1}^{\circ}(x)$, we obtain $Z_{K-2}^{\circ}(y) \leq \overline{Z}_{K-2}(y)$ (for all $y \in \mathbb{R}_+$). Proceeding this way, we end up with $Z_1^{\circ}(y) \leq \overline{Z}_1(y)$ (for all $y \in \mathbb{R}_+$). So, whenever the necessary conditions in (5) are satisfied also $Z_1^{\circ}(y) < 0$ holds for some $y \in [0, \overline{a}]$. The other way around, if $Z_1^{\circ}(y) \geq 0$ for all $y \in [0, \overline{a}]$ —which is equivalent with $G \succsim_{FHT} F$, basically because we can recursively replace $\min(-Z_k^{\circ})$ by $-\max Z_k^{\circ}$ in the expression for Z_1°— then the necessary conditions in (5) cannot be satisfied.

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